

Review of Discrete-Time Signals and Systems

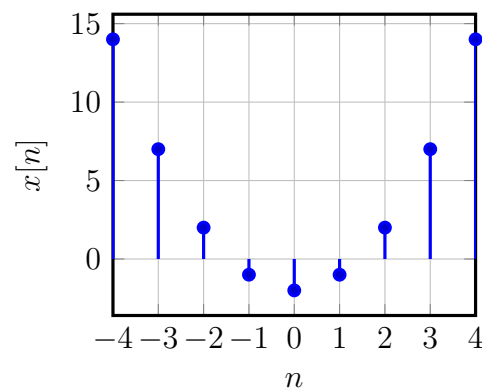
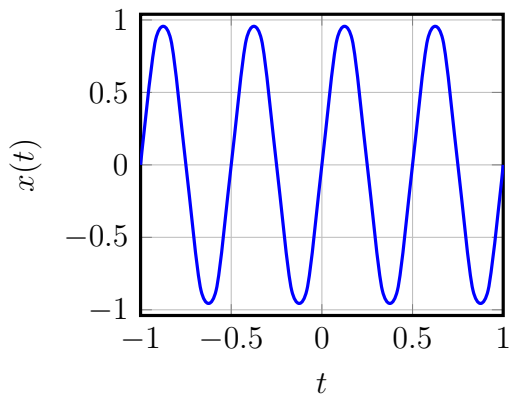
Henry D. Pfister
Based on Notes by Tie Liu

January 19, 2015

- Reading: A more detailed treatment of this material can be found in in Chapter 2 of Discrete-Time Signal Processing by Oppenheim and Schaffer or in Chapter 2 of Digital Signal Processing by Proakis and Manolakis (minus the DTFT).

1 Introduction

1.1 Signals



A *signal* is a function of an independent variable (e.g., time) that carries some information or describes some physical phenomenon.

- Notation
 - Continuous-time (CT) signal $x(t)$: independent variable t takes continuous values
 - Discrete-time (DT) signal $x[n]$: independent variable n takes only integer values
 - Note: $x(t)$ is used to denote both the “signal” and “the signal value at time t ”
- Examples

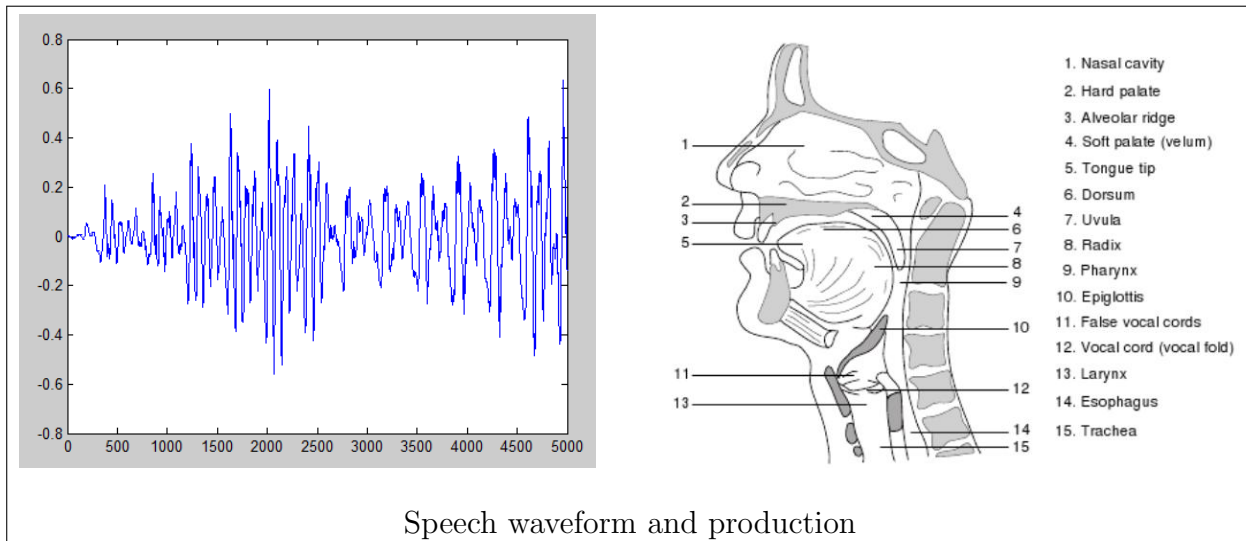
- Electrical signals: Voltages and currents in a circuit.
 - Acoustic signals: Audio and speech signals.
 - Biological signals: ECG, EEG, medical images.
 - Financial signals: Dow Jones indices.
- Independent variables
 - Can be continuous: Time and location.
 - Can be discrete: Digital image pixels, DNA sequence.
 - Can be 1-D, 2-D, ..., N -D.

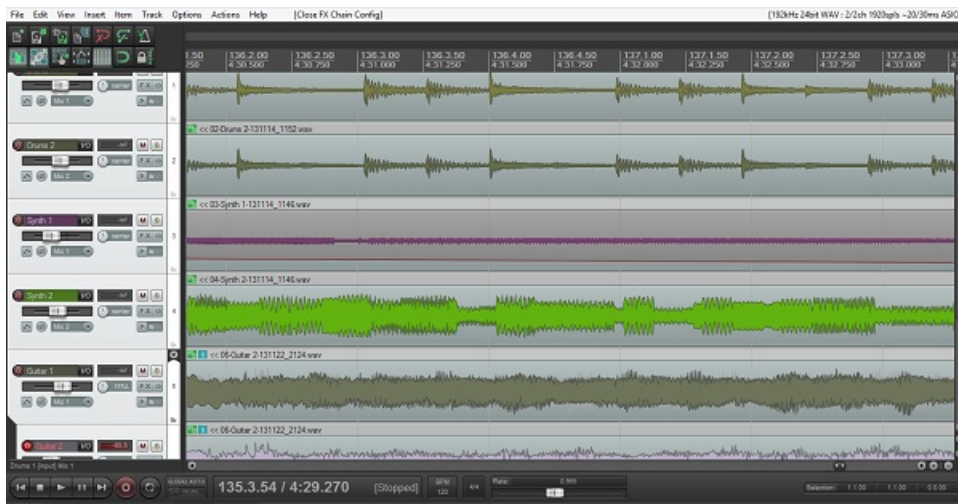
Most of the signals in the physical world are CT signals. DT signals are often formed by sampling a CT signal because DT signals can be directly processed by the powerful digital computers and digital signal processors (DSPs). This course focuses primarily on the digital processing of 1-D discrete-time audio signals.

1.2 Applications

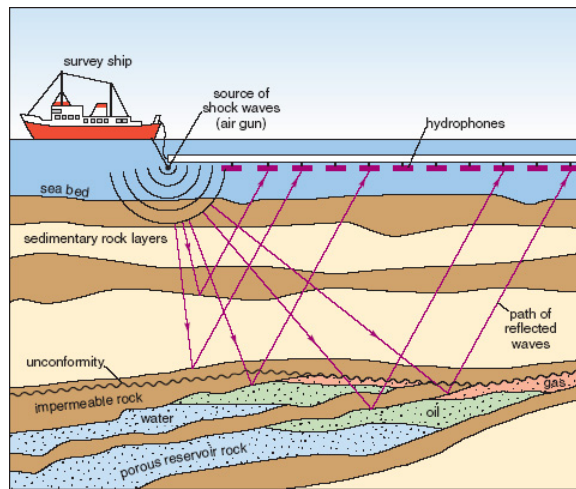
The analysis of signals and systems now plays a fundamental role in a wide range of engineering disciplines:

- Speech: recording, compression, synthesis
- Music: recording, processing, and synthesis
- Petroleum: Seismic surveying and Geological exploration
- Telecommunications: AM/FM Radio, speech, mobile phone, and internet

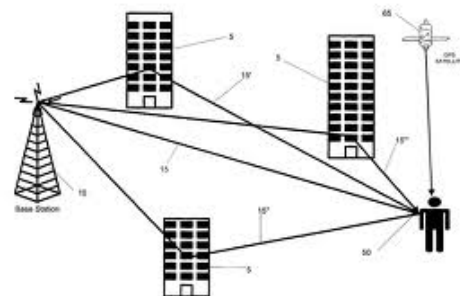
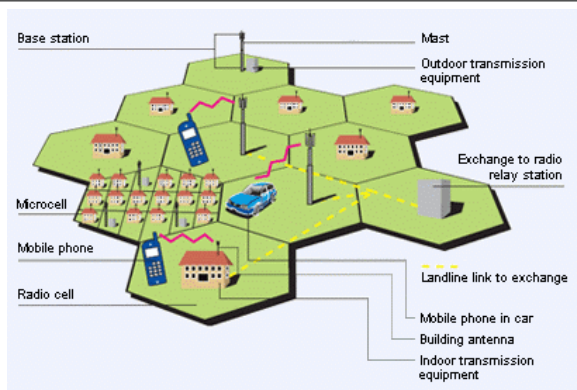




Digital Audio Workstation



Marine Reflection Seismology for Oil Exploration

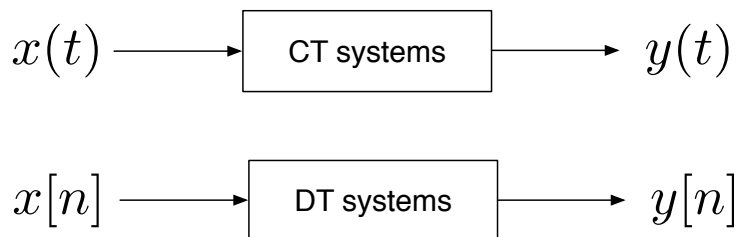


Wireless and Cellular Communication

1.3 Systems

A *system* responds to one or several input signals, and its response is described in terms of one or several output signals.

This course primarily focuses on single-input single-output (SISO) systems:



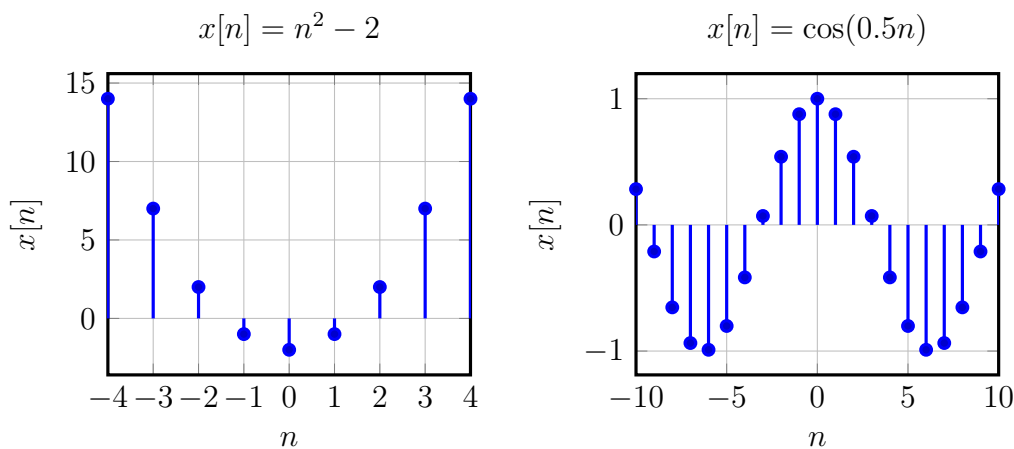
- Examples

- An RLC circuit.
- The dynamic of an aircraft.
- An edge detection algorithm for medical images.
- An algorithm for analyzing financial data to predict stock/bond prices.

Systems can be extremely diverse. However, from the input-output perspective, many systems share the same feature of being “linear” and “time-invariant”. The majority of the systems in this course will be linear time-invariant (LTI) systems.

2 Mathematical Description of Discrete-Time Signals

2.1 Definitions



2.2 Periodic signals

A DT signal $x[n]$ is *periodic with period N* if N is the smallest positive integer such that $x[n + N] = x[n]$ for all integer n . Likewise, induction implies that $x[n + kN] = x[n]$ for all integer k and all integer n . The following properties of periodic functions may be useful:

- Time-shifts remain periodic: $x[n]$ periodic with period N implies $x[n - n_0]$ is periodic with period N for all integer n_0 .
- Sums of periodic functions are periodic: if $x_1[n]$ is periodic with period N_1 and $x_2[n]$ is periodic with period N_2 , then $y[n] = x_1[n] + x_2[n]$ satisfies $y[n + \tau] = y[n]$ for all integer n with $\tau = \text{lcm}(N_1, N_2)$. This implies that τ is an integer multiple of the period, N , of $y[n]$.
- Functions of periodic functions are periodic: Actually, this statement holds for any function of M periodic signals with periods N_1, N_2, \dots, N_M by choosing $\tau = \text{lcm}(N_1, N_2, \dots, N_M)$.

2.3 Energy and Power

Let $x(t)$ be a continuous time signal. Suppose $v(t) = x(t)$ volts are applied across an R ohm resistor. Then, the instantaneous current is $i(t) = \frac{1}{R}v(t)$ amps and the instantaneous power dissipation is $p(t) = i(t)v(t) = \frac{1}{R}x(t)^2$ Watts. So, the total energy dissipated over the time interval $t_1 \leq t \leq t_2$ is given by

$$\int_{t_1}^{t_2} p(t)dt = \frac{1}{R} \int_{t_1}^{t_2} x(t)^2 dt.$$

and the average power over the same interval is

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} p(t)dt = \frac{1}{R(t_2 - t_1)} \int_{t_1}^{t_2} x(t)^2 dt.$$

With this simple model in mind, the standard convention is to define the energy and power of a signal as above with $R = 1$.

For complex signals, one must use $|x(t)|^2$ instead of $x(t)^2$. Also, for DT signals, the energy over the interval $n_1 \leq n \leq n_2$ is given by

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

and the average power is

$$\frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2.$$

The total energy in a signal (for CT and DT) is defined by

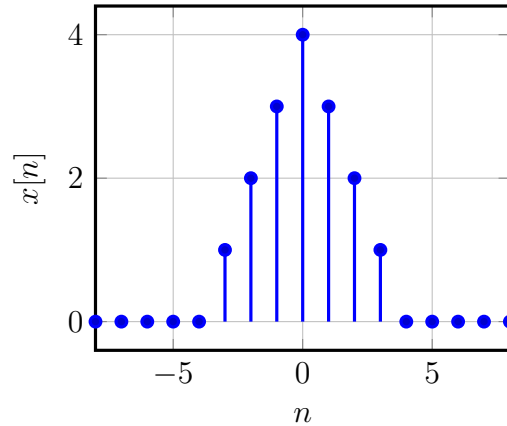
$$E_x = \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 \quad E_x = \lim_{N \rightarrow \infty} \sum_{n=-N}^N |x[n]|^2.$$

The average power of the whole signal (for CT and DT) is defined by

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 \quad P_x = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2.$$

Any signal with finite energy (i.e., $E_\infty < \infty$) has power $P_\infty = 0$ and is sometimes called an “energy-type” signal. Any signal with $0 < P_\infty < \infty$ has $E_\infty = \infty$ and is sometimes called a “power-type” signal.

$$x[n] = \begin{cases} 4 \left(1 - \frac{n}{4}\right) & \text{if } 0 < n \leq 4 \\ 4 \left(1 + \frac{n}{4}\right) & \text{if } -4 < n \leq 0 \\ 0 & \text{otherwise} \end{cases}$$



Example 1. What is the energy of $x[n]$? Summing gives

$$E_x = \sum_{n=-3}^3 (4 - |n|)^2 = 2(1 + 4 + 9) + 16 = 44.$$

2.4 Correlation

For DT energy-type signals, the *cross correlation* between $x[n]$ and $y[n]$ with lag ℓ is defined to be

$$r_{xy}[\ell] \triangleq \sum_{n=-\infty}^{\infty} x[n]y^*[n - \ell] = \sum_{n=-\infty}^{\infty} x[n + \ell]y^*[n].$$

This quantity measures how closely the two signals match each other when they are shifted and scaled. In particular, the squared Euclidean distance satisfies

$$\sum_{n=-\infty}^{\infty} |x[n] - Ay[n - \ell]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2 + |A|^2 \sum_{n=-\infty}^{\infty} |y[n - \ell]|^2$$

$$\begin{aligned}
& - \sum_{n=-\infty}^{\infty} (A^* x[n] y^*[n - \ell] + A x^*[n] y[n - \ell]) \\
& = E_x + |A|^2 E_y - 2 \operatorname{Re} \{ A r_{xy}^*[\ell] \}.
\end{aligned}$$

Moreover, the minimum¹ over A equals $E_x - |r_{xy}[\ell]|^2/E_y$ and is achieved by $A = r_{xy}[\ell]/E_y$.

This operation is very useful if one is trying to find a scaled and shifted copy of one signal as a component of another. For example, consider the case where $y[n] = Bx[n - n_0]$ for a known $x[n]$ and unknown parameters B and n_0 . The autocorrelation can be used to compute the parameters.

The *autocorrelation* $r_{xx}[\ell]$ is defined to be the cross-correlation between $x[n]$ and itself (i.e., $r_{xy}[\ell]$ when $y[n] = x[n]$). The maximum autocorrelation is always $r_{xx}[0] = E_x$ and, hence, the *normalized autocorrelation* is defined to be $r_{xx}[\ell]/r_{xx}[0]$. Also, autocorrelation of a periodic signal with period N will take its maximum value of E_x when ℓ is an integer multiple of N . Thus, the autocorrelation is very useful for detecting periodicity in a signal. Also, if $y[n] = x[n] + Bx[n - n_0]$ has an echo with time delay n_0 , then autocorrelation can be used to estimate B and n_0 .

For power-type signals, similar results hold for the *time-average cross-correlation* function

$$\bar{r}_{xy}[\ell] \triangleq \lim_{M \rightarrow \infty} \frac{1}{2M+1} \sum_{n=-M}^M x[n] y^*[n - \ell] = \lim_{M \rightarrow \infty} \frac{1}{2M+1} \sum_{n=-M}^M x[n + \ell] y^*[n].$$

Problem 1 (DSP-4 2.63). What is the normalized autocorrelation sequence of the signal $x(n)$ given by

$$x(n) = \begin{cases} 1 & \text{if } -N \leq n \leq N \\ 0 & \text{otherwise.} \end{cases}$$

2.5 Transformation of Discrete-Time Signals

There are many ways of transforming a DT signal into another. One can scale it by multiplying by a constant. One can time-shift it by adding a delay. One can even add up scaled and time-shifted copies of the signal.

First, let us consider transformations (i.e., systems) of the form:

$$x[n] \rightarrow y[n] = x[f[n]]$$

where $x[n]$ is the input signal, $y[n]$ is the output signal, and $f[n]$ is an integer signal. The arrow “ \rightarrow ” denotes the action and direction of transformation. The function $f[n]$ can be an arbitrary integer function but we will first consider the class of affine functions

$$f[n] = an + b$$

¹This is found by separately computing the derivatives with respect to $\operatorname{Re}\{A\}$ and $\operatorname{Im}\{A\}$.

where $a \neq 0$ and b are integers. All affine transformations can be decomposed into just three fundamental types of signal transformations on the independent variable: time shift, time scaling, and time reversal. They involve a change of the variable t into something else:

- Time shift: $f[n] = n - n_0$ for some $n_0 \in \mathbb{Z}$.
- Time scaling: $f[n] = an$ for some $a \in \mathbb{Z}^+$.
- Time reversal (or flip): $f[n] = -n$.

Transformations in discrete time are analogous to those in continuous time. However, there are a few subtle points to consider. For instance, can we time shift $x[n]$ by a non-integer delay, say to $x[n - 1/2]$? If we compress the signal $x[n]$ to $x[2n]$, do we lose half the information stored? Finally, if we expand $x[n]$ to $x[n/2]$, how do we “fill in the blanks?” These questions will be addressed later when we consider interpolation and decimation.

3 System Properties

3.1 Linearity

A DT system $x[n] \rightarrow y[n]$ is *linear* if it satisfies:

$$x_1[n] \rightarrow y_1[n] \quad \text{and} \quad x_2[n] \rightarrow y_2[n] \quad \implies \quad ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n], \quad \forall a, b \in \mathbb{C}$$

A linear system satisfies the *superposition* property:

$$x_k[n] \rightarrow y_k[n] \quad \forall k \implies \sum_k a_k x_k[n] \rightarrow \sum_k a_k y_k[n], \quad \forall a_k \in \mathbb{C}$$

Example 2. $x[n] \rightarrow y[n] = x[n] + x[n - 1]$ is linear but $x[n] \rightarrow y[n] = x[n]x[n - 1]$ is not linear.

3.2 Time-Invariance

Informally, a system is *time-invariant (TI)* if its behavior does not depend on what time it is. Mathematically, a DT system $x[n] \rightarrow y[n]$ is TI if

$$x[n] \rightarrow y[n] \quad \implies \quad x[n - n_0] \rightarrow y[n - n_0], \quad \forall n_0 \in \mathbb{Z}$$

Example 3. Consider $x[n] \rightarrow y[n] = x^2[n - 1]$. To check whether this system is TI or time varying (TV), we need to determine the output corresponding to the input $x[n - n_0]$ and then compare it with $y[n - n_0]$. Let $x'[n] = x[n - n_0]$. Then $y'[n] = [x'[n - 1]]^2 = [x[n - 1 - n_0]]^2$. On the other hand, $y[n - n_0] = x^2[n - n_0 - 1]$. Thus, $y'[n] = y[n - n_0]$ and the system is TI. What about the system $x[n] \rightarrow y[n] = x[-n]$?

Proposition 4. *If the input to a TI system is periodic with a period N , then the output is also periodic with a period N .*

Proof. Suppose $x[n+N] = x[n]$ for all $n \in \mathbb{Z}$ and $x[n] \rightarrow y[n]$. Then, by TI, $x[n+N] \rightarrow y[n+N]$. Since the output of a system is determined by its input, it follows that $y[n] = y[n+N]$. In other words, the output is also periodic with period N . \square

4 Linear Time-Invariant Systems

Recall that a DT system is time invariant (TI) if

$$x[n] \rightarrow y[n] \implies x[n - n_0] \rightarrow y[n - n_0], \quad \text{for all integer } n_0$$

and that it is linear if

$$x_1[n] \rightarrow y_1[n] \quad \text{and} \quad x_2[n] \rightarrow y_2[n] \implies ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n], \quad \text{for all complex } a, b$$

For this class, we focus on systems that are both linear and time invariant (LTI) due to:

- practical importance and
- the mathematical tools available for the analysis of LTI systems.

A basic fact: If we know the response of an LTI system to some inputs, then we actually know the response to many inputs.

Question: What is the *smallest* set of inputs for which, if we know their outputs, we can determine the output of *any* input signal?

4.1 Discrete-Time Convolution

For DT systems, the answer is surprisingly simple: All we need to know is the *impulse response* (denoted by $h[n]$) which is the response to a unit impulse input

$$\delta[n] \triangleq \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

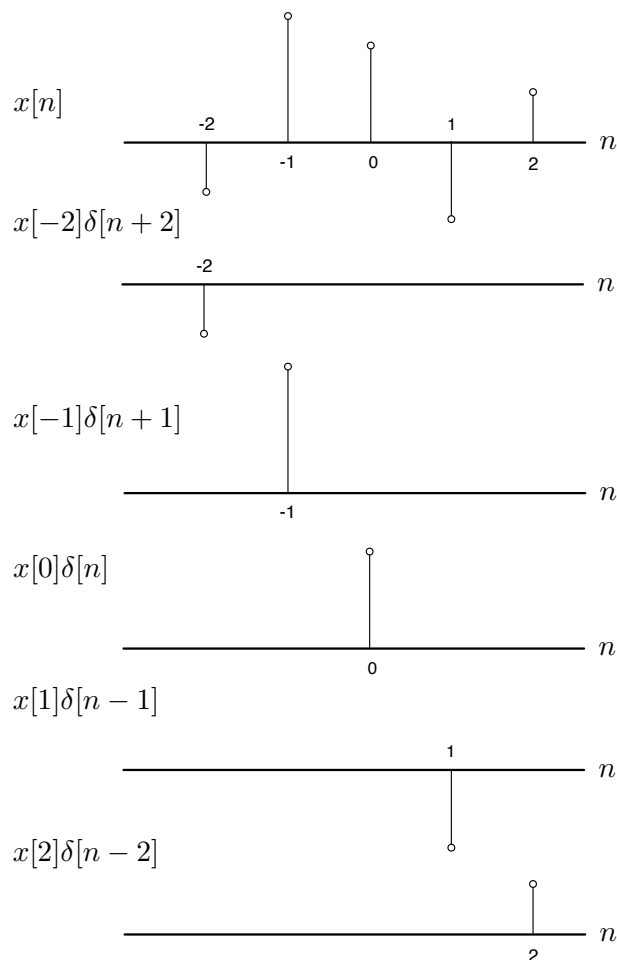
As an aside, we also define here the *unit step function*

$$u[n] \triangleq \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0. \end{cases}$$

The reason one only needs the impulse response is that we can write any signal $x[n]$ as a linear combination of the unit impulse function and its time-shifts:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

where $x[k]$ are coefficients and $\delta[n-k]$ is a time shift of $\delta[n]$. Mathematically, this is equivalent to noting that the canonical unit vectors (i.e., $\{\delta[n-k]\}_{k \in \mathbb{Z}}$) form a basis for the space of complex sequences with bounded entries (i.e., ℓ_{∞}).



Let $h_k[n]$ be the system response to the input $\delta[n-k]$. By linearity,

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

Furthermore, by TI,

$$\delta[n] \rightarrow h[n] \implies \delta[n-k] \rightarrow h_k[n] = h[n-k]$$

The surprising conclusion is that the output of an LTI system is given by the “convolution” sum

$$y[n] = x[n] * h[n] \triangleq \sum_{k=-\infty}^{\infty} x[k]h[n - k]$$

Observation: If we know the unit impulse response $h[n]$ of a LTI system, we can compute the output $y[n]$ of an arbitrary input $x[n]$ as $y[n] = x[n] * h[n]$. In this sense, a LTI system is fully determined by its unit impulse response.

Visualizing the calculation of convolution sum:

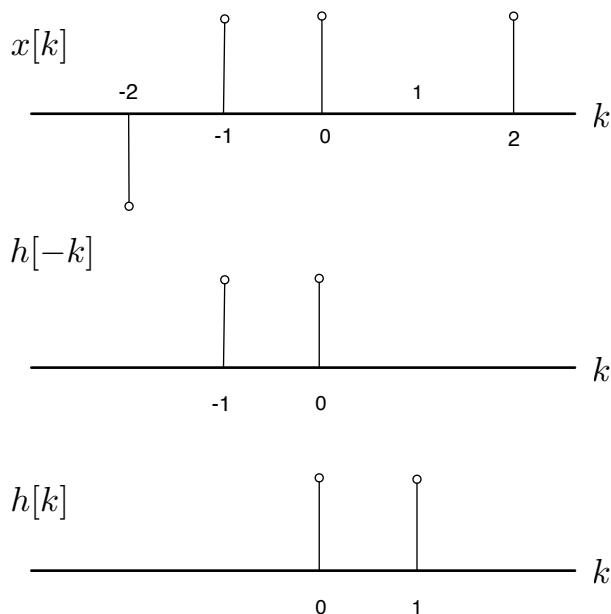
Step 1: Choose a value of n and consider it fixed.

Step 2: Plot $x[k]$ as a function of k .

Step 3: Plot the function $h[n - k]$ (as a function of k) by first flipping $h[k]$ and then shift to the right by n (if n is negative, this means a shift to the left by $|n|$).

Step 4: Compute the intermediate signal $w_n[k] \triangleq x[k]h[n - k]$ via pointwise multiplication and then sum this signal to obtain the result $y[n]$.

To compute $y[n + 1]$, one can compute $h[n + 1 - k]$ simply by shifting $h[n - k]$ to the right by sample. Then, answer is computed by repeating Step 4.



Problem 2 (DSP-4 2.20). Consider the following three operations:

- (a) Multiply the integer numbers: 131 and 122.

- (b) Compute the convolution of the signals: $\{1, 3, 1\} * \{1, 2, 2\}$.
- (c) Multiply the polynomials: $1 + 3z + z^2$ and $1 + 2z + 2z^2$.
- (d) Repeat part (a) for the numbers 1.31 and 12.2.
- (e) Comment on your results.

Problem 3 (DSP-4 2.54). Compute and sketch the convolution $y_i(n)$ and correlation $r_i(n)$ sequences for the following pairs of signals and comment on the results obtained.

- (a) $x_1(n) = \{\underset{\uparrow}{1}, 2, 4\}$ $h_1(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 1\}$
- (b) $x_2(n) = \{\underset{\uparrow}{0}, 1, -2, 3, -4\}$ $h_2(n) = \{\frac{1}{2}, 1, \underset{\uparrow}{2}, 1, \frac{1}{2}\}$
- (c) $x_3(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$ $h_3(n) = \{\underset{\uparrow}{4}, 3, 2, 1\}$
- (d) $x_4(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$ $h_4(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$

Problem 4 (DSP-4 2.22). Let $x(n)$ be the input signal to a discrete-time filter with impulse response $h_i(n)$ and let $y_i(n)$ be the corresponding output.

- (a) Compute and sketch $x(n)$ and $y_i(n)$ for the following, using the same scale in all figures.

$$\begin{aligned}
 x(n) &= \{1, 4, 2, 3, 5, 3, 3, 4, 5, 7, 6, 9\} \\
 h_1(n) &= \{1, 1\} \\
 h_2(n) &= \{1, 2, 1\} \\
 h_3(n) &= \{\frac{1}{2}, \frac{1}{2}\} \\
 h_4(n) &= \{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\} \\
 h_5(n) &= \{\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\}
 \end{aligned}$$

Sketch $x(n), y_1(n), y_2(n)$ on one graph and $x(n), y_3(n), y_5(n)$ on another graph.

- (b) What is the difference between $y_1(n)$ and $y_2(n)$, and between $y_3(n)$ and $y_4(n)$?
- (c) Comment on the smoothness of $y_2(n)$ and $y_4(n)$. Which factors affect the smoothness?
- (d) Compare $y_4(n)$ with $y_5(n)$. What is the difference? Can you explain it?
- (e) Let $h_6(n) = \{\frac{1}{2}, -\frac{1}{2}\}$. Compute $y_6(n)$. Sketch $x(n), y_2(n)$ and $y_6(n)$ on the same figure and comment on the results.

Next, we consider the discrete analog of linear constant-coefficient differential equations.

Definition 5. A *linear constant-coefficient difference equation* (LCCDE),

$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m],$$

defines the relationship between an input sequence $x[n]$ and an output sequence $y[n]$. If an LTI system is causal, then it can be described by a LCCDE.

Example 6. Consider the DT system described by the LCCDE

$$y[n] - \frac{1}{2}y[n-1] = x[n].$$

We assume the system is initially at rest (i.e., causal), which is defined mathematically as

$$x[k] = 0 \text{ for all integer } k \leq k_0 \quad \implies \quad y[k] = 0 \text{ for all integer } k \leq k_0.$$

It turns out that a system is LTI if it is described by a LCCDE and it is initially at rest. In this case, we can first figure out the unit impulse response of the system $h[n]$ and then use the convolution sum to calculate the response to $u[n]$. It is not too hard to verify that $y[n] = \left(\frac{1}{2}\right)^n u[n]$ satisfies the above LCCDE for input $x[n] = \delta[n]$. Therefore, we say that its impulse response is

$$h[n] = \left(\frac{1}{2}\right)^n u[n].$$

Now, we can calculate the output associated with the input $x[n] = u[n]$, which is known as the *step response* of the system. By the convolution sum, the output $y[n]$ corresponding to the input $x[n] = u[n]$ is given by

$$\begin{aligned} y[n] &= u[n] * h[n] \\ &= \left(\sum_{k=0}^{\infty} \delta[n-k] \right) * h[n] \\ &= \sum_{k=0}^{\infty} h[n-k]. \end{aligned}$$

Thus, we find that

$$\begin{aligned} y[0] &= 1 \\ y[1] &= \frac{1}{2} + 1 = \frac{3}{2} \\ y[2] &= \left(\frac{1}{2}\right)^2 + \frac{1}{2} + 1 = \frac{7}{4} \\ &\vdots \end{aligned}$$

$$y[n] = \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} + \dots + \frac{1}{2} + 1 = \frac{1 - \frac{1}{2}\left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 2 - \left(\frac{1}{2}\right)^n$$

for $n \geq 0$ and $h[n] = 0$ for $n \leq -1$.

Problem 5 (DTSP-2 2.31). Consider the difference equation

$$y(n) + \frac{1}{15}y(n-1) - \frac{2}{15}y(n-2) = x(n).$$

- Determine the general form of the homogeneous solution to this equation.
- Both a causal and an anticausal LTI system are characterized by the given difference equation. Find the impulse responses of the two systems.
- Show that the causal LTI system is stable and the anticausal LTI system is unstable.
- Find a particular solution to the difference equation when $x(n) = \left(\frac{3}{5}\right)^n u(n)$.

5 Properties of Convolution

Definition 7 (The sifting property). Convolution with a time-shifted impulse results in a time-shifted version of the output:

$$x[n] * \delta[n - n_0] = x[n - n_0].$$

Therefore, $x[n] * \delta[n] = x[n]$ and $\delta[n]$ is the identity element under convolution.

Proof. By definition,

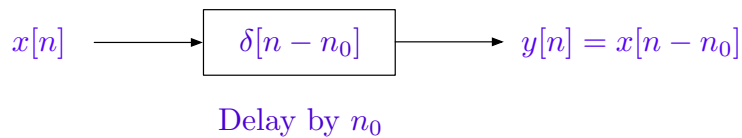
$$x[n] * \delta[n - n_0] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k - n_0]$$

and $x[k] \delta[n - k - n_0] = x[n - n_0] \delta[n - k - n_0]$ because

$$\delta[n - k - n_0] = \begin{cases} 1, & k = n - n_0 \\ 0, & k \neq n - n_0 \end{cases}$$

Computing the sum shows that $y[n] = x[n - n_0]$ and therefore $x[n] * \delta[n - n_0] = x[n - n_0]$. \square

We can interpret this property using the following block diagram:



Definition 8 (The commutative property). The convolution operator is commutative:

$$y[n] = x[n] * h[n] = h[n] * x[n].$$

Proof. By definition,

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

and

$$h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

For the first, sum we can substitute $k = n - k'$ to get

$$x[n] * h[n] = \sum_{k'=-\infty}^{\infty} x[n-k']h[k'],$$

where the limits of summation are unchanged because they are infinite. We conclude that $x[n] * h[n] = h[n] * x[n]$. □

Definition 9 (The distributive property). Convolution distributes over addition:

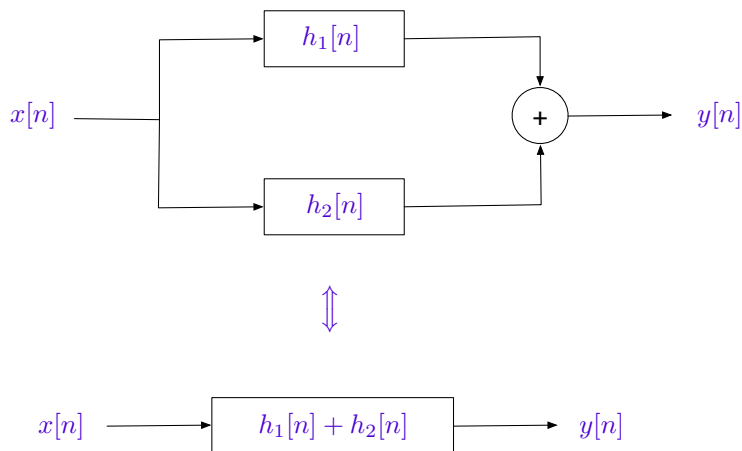
$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

Proof. By definition,

$$\begin{aligned} x[n] * (h_1[n] + h_2[n]) &= \sum_{k=-\infty}^{\infty} x[k](h_1[n-k] + h_2[n-k]) \\ &= \sum_{k=-\infty}^{\infty} x[k]h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k]h_2[n-k] \\ &= x[n] * h_1[n] + x[n] * h_2[n] \end{aligned}$$

□

Interpretation:



Definition 10 (The associative property). $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$

Proof. Let

$$a[n] = h_1[n] * h_2[n] = \sum_{k=-\infty}^{\infty} h_1[k]h_2[n-k]$$

and

$$b[n] = x[n] * h_1[n] = \sum_{l=-\infty}^{\infty} x[l]h_1[n-l]$$

Then

$$\begin{aligned} x[n] * a[n] &= \sum_{l=-\infty}^{\infty} x[l]a[n-l] \\ &= \sum_{l=-\infty}^{\infty} x[l] \left(\sum_{k=-\infty}^{\infty} h_1[k]h_2[n-l-k] \right) \end{aligned}$$

and

$$\begin{aligned} b[n] * h_2[n] &= \sum_{k=-\infty}^{\infty} b[k]h_2[n-k] \\ &= \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{\infty} x[l]h_1[k-l] \right) h_2[n-k] \\ &= \sum_{l=-\infty}^{\infty} x[l] \left(\sum_{k=-\infty}^{\infty} h_1[k-l]h_2[n-k] \right) \end{aligned}$$

Let $k' = k - l$. We have $k = l + k'$ and hence

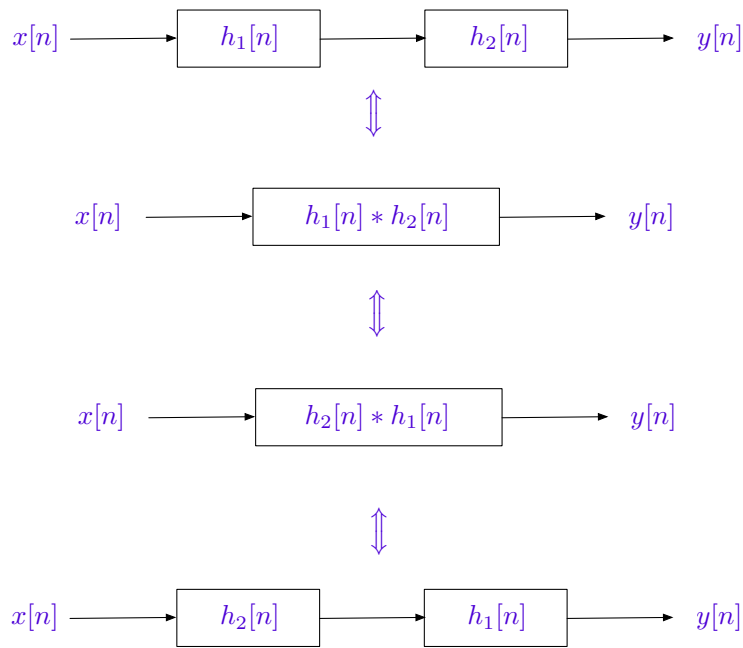
$$b[n] * h_2[n] = \sum_{l=-\infty}^{\infty} x[l] \left(\sum_{k'=-\infty}^{\infty} h_1[k']h_2[n-l-k'] \right)$$

Note that k and k' are indices of summation and, therefore, we conclude that $x[n] * a[n] = b[n] * h_2[n]$. \square

Combining properties 2 and 4, we have

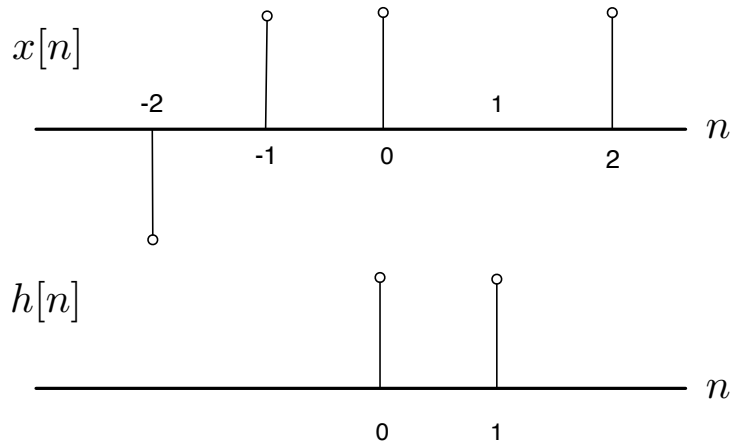
$$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n]) = x[n] * (h_2[n] * h_1[n]) = (x[n] * h_2[n]) * h_1[n]$$

Interpretation:



Combining properties 1 and 3, one can compute the convolution of two finite signals algebraically.

Example 11. Consider the signals



and observe that

$$x[n] = -\delta[n + 2] + \delta[n + 1] + \delta[n] + \delta[n - 2]$$

and

$$h[n] = \delta[n] + \delta[n - 1].$$

Thus,

$$x[n] * h[n] = (-\delta[n + 2] + \delta[n + 1] + \delta[n] + \delta[n - 2]) * (\delta[n] + \delta[n - 1])$$

$$\begin{aligned}
&= -\delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-2] + \\
&\quad (-\delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-3]) \\
&= -\delta[n+2] + 2\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]
\end{aligned}$$

Definition 12 (Causality). A DT LTI system is causal if and only if its unit impulse response $h[n] = 0$ for all $n < 0$.

Proof. A linear system $x[n] \rightarrow y[n]$ is causal if and only if, for all $k \in \mathbb{Z}$, $x[n] = 0$ for $n < k$ implies $y[n] = 0$ for $n < k$. If a DT LTI system is causal, then the unit impulse response $h[n] = 0$ for all $n < 0$, because its input $\delta[n] = 0$ for all $n < 0$. Conversely, if the unit impulse response $h[n] = 0$ for all $n < 0$, by the convolution sum

$$y[n] = x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k \geq 0} h[k]x[n-k]$$

i.e., $y[n]$ is fully determined by the current and previous inputs $x[n-k]$ for $k \geq 0$. We conclude that the system must be causal. \square

Definition 13 (Stability). A DT LTI system is stable if and only if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$.

Proof. A system is stable if the output is bounded whenever the input is bounded. For an LTI system, if $A = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$ and the input $x[n]$ is bounded (i.e., $|x[n]| \leq M_x < \infty$ for all n), then

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |x[k]||h[n-k]| \leq M_x \sum_{k=-\infty}^{\infty} |h[n-k]| = M_x A < \infty.$$

Thus, the output is bounded and the system is stable. Conversely, if $A = \infty$, then the output $y[n]$ associated with input $x[k] = \text{sgn}(h[-k])$ is not bounded because $y[0] = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$. \square

Problem 6 (DSP-4 2.7). A discrete-time system can be: (1) Static or dynamic, (2) Linear or non-linear, (3) Time invariant or time varying, (4) Causal or non-causal, (5) Stable or unstable. Examine the following systems with respect to the properties above:

- (a) $y(n) = \cos[x(n)]$.
- (b) $y(n) = \sum_{k=-\infty}^{n+1} x(k)$.
- (c) $y(n) = x(n) \cos(\omega_0 n)$.
- (d) $y(n) = x(n) + nx(n+1)$.

(e) $y(n) = x(n)u(n)$.

Problem 7 (DSP-4 2.25). Consider the signal $\gamma(n) = a^n u(n)$, $0 < a < 1$.

(a) Show that any sequence $x(n)$ can be decomposed as

$$x(n) = \sum_{k=-\infty}^{\infty} c_k \gamma(n-k)$$

and express c_k in terms of $x(n)$. Hint: Consider $\gamma(n) - a\gamma(n-1)$.

(b) Use the properties of linearity and time invariance to express the output $y(n) = \mathcal{T}[x(n)]$ in terms of the input $x(n)$ and the signal $g(n) = \mathcal{T}[\gamma(n)]$, where $\mathcal{T}[\cdot]$ is an LTI system.

(c) Express the impulse response $h(n) = \mathcal{T}[\delta(n)]$ in terms of $g(n)$.

A final word: We said all along that a DT LTI system is fully characterized by its unit impulse response $h[n]$. However, if a system is not LTI to begin with, knowing the unit impulse response $h[n]$ may shed very little insight about what the system is. In fact, many systems may share the same unit impulse response $h[n]$. Consider, for example, $h[n] = \delta[n]$ is the unit impulse response to any system $y[n] = (x[n])^k$ with an integer k . But among them, there is only one system that is LTI. That is, $y[n] = x[n]$.

6 The Eigenfunctions of LTI Systems

6.1 Motivating Example

Consider the LTI system $y[n] = x[n] + x[n-2]$ with the input $\cos(\omega n + \phi)$. What can we say about the output in this case? Using the identity $\cos(a) + \cos(b) = 2 \cos\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right)$, one finds that

$$\begin{aligned} y[n] &= \cos(\omega n + \phi) + \cos(\omega(n-2) + \phi) \\ &= 2 \cos(\omega) \cos(\omega n + (\phi - \omega)). \end{aligned}$$

The result is that we get an output waveform of the same frequency, but with a phase shift and an amplitude scaling. Therefore, the input-output mapping for sinusoidal inputs only consists of a scale factor and phase shift. It is not too hard to generalize this and show that the same thing happens for arbitrary LTI systems! In the next section, we will use the power of complex numbers to derive stronger results with even simpler arguments.

6.2 Complex Exponentials

Consider the DT complex exponential input $x[n] = z^n$, where $z = re^{j\Omega}$ is an arbitrary complex number. The exponential growth rate of this signal is determined by r and the oscillation period is given by $N = \text{lcm}(1, 2\pi/\Omega)$ (i.e., the smallest multiple of $2\pi/\Omega$ that is also integer). If $2\pi/\Omega$ is not a rational number, then $N = \infty$ and the signal is not periodic. For a DT LTI system with impulse response $h[n]$, the associated output is given by the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \underbrace{\sum_{k=-\infty}^{\infty} h[k]z^{-k}}_{H(z)}.$$

This implies that the output signal is equal to the input signal multiplied by the complex constant $H(z)$, which depends explicitly on z and implicitly on the system impulse response $h[n]$. The function $H(z)$ is called the *transfer function* of the LTI system.

In linear algebra, an *eigenvector* of a linear transform is an input whose output equals the input multiplied by a constant. For example, a vector v is an eigenvector of a matrix A if $Av = \lambda v$ for some scalar λ . Since we view a signal $x[n]$ as a function of n , we say that the complex exponential $x[n] = z^n$ is an *eigenfunction* of a DT LTI system because this input produces the input multiplied the eigenvalue $H(z)$ for any DT LTI system.

Example 14. For an arbitrary LTI system, consider again the input

$$x[n] = \cos(\omega n + \phi) = \frac{1}{2} (e^{j(\omega n + \phi)} + e^{-j(\omega n + \phi)})$$

and observe that

$$y[n] = \frac{1}{2} e^{j\phi} H(e^{j\omega}) e^{j\omega n} + \frac{1}{2} e^{-j\phi} H(e^{-j\omega}) e^{-j\omega n}.$$

If $h[n]$ is real, then the transfer function has conjugate symmetry:

$$\begin{aligned} (H(e^{j\omega}))^* &= \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right)^* \\ &= \sum_{k=-\infty}^{\infty} h^*[k] e^{j\omega k} \\ &= \sum_{k=-\infty}^{\infty} h[k] e^{-(-j\omega)k} \\ &= H(e^{-j\omega}). \end{aligned}$$

Thus, we find that the output is given by

$$\begin{aligned} y[n] &= (e^{j\phi} H(e^{j\omega})) \frac{1}{2} e^{j\omega n} + (e^{j\phi} H(e^{j\omega}))^* \frac{1}{2} e^{-j\omega n} \\ &= |e^{j\phi} H(e^{j\omega})| \frac{1}{2} (e^{j(\phi+\theta)} e^{j\omega n} + e^{-j(\phi+\theta)} e^{-j\omega n}) \\ &= |e^{j\phi} H(e^{j\omega})| \cos(\omega n + \phi + \theta), \end{aligned}$$

where $\theta = \angle H(e^{j\omega})$ is the phase of the complex number $H(e^{j\omega})$. Thus, for a sinusoidal input, an LTI system preserves the frequency and only affects the magnitude and phase.

Example 15. Consider the LTI system $x[n] \rightarrow y[n] = x[n] + 0.5x[n-1]$ with impulse response $h[n] = \delta[n] + 0.5\delta[n-1]$. For this system, the transfer function is given by

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k} = \sum_{k=-\infty}^{\infty} (\delta[k] + 0.5\delta[k-1])z^{-k} = 1 + 0.5z^{-1}.$$

Therefore, the above analysis shows that the input $x[n] = 0.25^n$ will produce the output

$$y[n] = H(0.25)x[n] = (1 + 0.5/0.25)x[n] = 3(0.25)^n.$$

In this simple example, this input-output relationship might also be observed directly

$$y[n] = x[n] + 0.5x[n-1] = (0.25)^n + 0.5(0.25)^{n-1} = (1 + 0.5/0.25)(0.25)^n = 3(0.25)^n.$$

For complicated problems, however, we will find that the transfer function approach is very powerful.

Problem 8 (DTSP-2 2.13). Indicate which of the following discrete-time signals are eigenfunctions of discrete-time systems that are stable and linear time-invariant:

- | | | |
|------------------------|-----------------------|--|
| (a) $e^{j2\pi n/3}$ | (b) 3^n | (c) $2^n u(-n-1)$ |
| (d) $\cos(\omega_0 n)$ | (e) $(\frac{1}{4})^n$ | (f) $(\frac{1}{4})^n u(n) + 4^n u(-n-1)$ |

6.3 Connection in Series

Now, consider the LTI system, with impulse response $h[n]$, defined by the series connection of two LTI systems, with impulse responses $h_1[n]$ and $h_2[n]$. Let the input signal be $x[n] = z^n$, the output of the first system be $w[n]$, and the overall output be $y[n]$. This implies that $w[n] = H_1(z)x[n]$, where the transfer function of the first system is

$$H_1(z) = \sum_{k=-\infty}^{\infty} h_1[k]z^{-k}.$$

Likewise, $y[n] = H_2(z)w[n]$, where the transfer function of the second system is

$$H_2(z) = \sum_{k=-\infty}^{\infty} h_2[k]z^{-k}.$$

Therefore, the output is given by $y[n] = H_1(z)H_2(z)z^n$ and we see the transfer function $H(z) = H_1(z)H_2(z)$ of the overall system is given simply by the product of the individual transfer functions.

Problem 9 (DSP-4 2.50). Consider the discrete-time system shown in Fig. 1:

- Compute the first six values of the impulse response of the system.
- Compute the first six values of the zero-state step response of the system.
- Determine an analytical expression for the impulse response of the system.

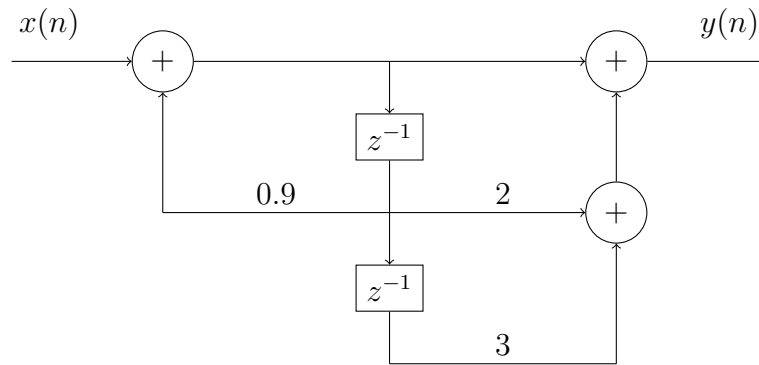


Figure 1: Discrete-time system for Problem DSP-4 2.50.

7 Discrete-Time Fourier Transform

7.1 Definition

The discrete-time Fourier transform (DTFT) maps an aperiodic discrete-time signal $x[n]$ to the frequency-domain function $X(e^{j\Omega}) = \mathcal{F}\{x[n]\}$. Likewise, we write $x[n] = \mathcal{F}^{-1}\{X(e^{j\Omega})\}$.

The DTFT pair $x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\Omega})$ satisfies:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega \quad (\text{synthesis equation})$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \quad (\text{analysis equation})$$

Remarks:

- Note that

$$e^{j(\Omega+2\pi)n} = e^{j\Omega n} e^{j2\pi n} = e^{j\Omega n}$$

for any integer n . So, in the synthesis equation, $\int_{2\pi}$ represents integration over any interval $[a, a + 2\pi)$ of length 2π . On the other hand, $x[n]$ is assumed to be aperiodic, so the summation in the analysis equation is over all n .

- Thus, the DTFT $X(e^{j\Omega})$ is always periodic with period 2π .
- The frequency response of a DT LTI system with unit impulse response $h[n]$

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\Omega n}$$

is precisely the DTFT of $h[n]$, so the response $y[n]$ that corresponds to the input $x[n] = e^{j\Omega n}$ is given by

$$y[n] = H(e^{j\Omega}) e^{j\Omega n}.$$

Convergence. If $x[n]$ is absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

then $X(e^{j\Omega})$ is well-defined (i.e., finite) for all $\Omega \in \mathbb{R}$. Thus, there is a unique $X(e^{j\Omega})$ for each absolutely summable $x[n]$.

Inversion. If $X(e^{j\Omega})$ is the DTFT of an absolutely summable DT signal $x[n]$, then

$$\begin{aligned} \frac{1}{2\pi} \int_{2\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega &= \frac{1}{2\pi} \int_{2\pi} \left(\sum_{m=-\infty}^{\infty} x[m] e^{-j\Omega m} \right) e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} x[m] \underbrace{\int_{2\pi} e^{-j\Omega(n-m)} d\Omega}_{2\pi\delta[n-m]} \\ &= \sum_{m=-\infty}^{\infty} x[m] \delta[n-m] \\ &= x[n]. \end{aligned}$$

Thus, there is a one-to-one correspondence between an absolutely summable $x[n]$ and its DTFT $X(e^{j\Omega})$.

Periodic Signals. For a periodic DT signal, $x[n] = x[n + N]$, let a_k be the discrete-time Fourier series coefficients

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_0 n},$$

where $\Omega_0 = 2\pi/N$. Then, the DTFT consists is a sum of Dirac delta functions:

$$X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0).$$

7.2 DTFT Examples

There are a few important DTFT pairs that are used regularly.

Example 16. Consider the *discrete-time rectangular pulse*

$$x[n] = \sum_{k=0}^M \delta[n - k].$$

The DTFT of this signal is given by

$$\begin{aligned} X(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \\ &= \sum_{k=0}^M e^{-j\Omega k} \\ &= \frac{e^{-j\Omega(M+1)} - 1}{e^{-j\Omega} - 1} \\ &= \frac{(e^{-j\Omega(M+1)/2} - e^{j\Omega(M+1)/2}) e^{-j\Omega(M+1)/2}}{(e^{-j\Omega/2} - e^{j\Omega/2}) e^{-j\Omega/2}} \\ &= \frac{\sin(\Omega(M+1)/2)}{\sin(\Omega/2)} e^{-j\Omega M/2}. \end{aligned}$$

The first term is the discrete-time analog of the sinc function and the second term is the phase shift associated with the fact that the pulse is not centered about $n = 0$. If M is even, we can advance the pulse by $M/2$ samples to remove this term.

Example 17. An *ideal discrete-time low-pass filter* passing $|\Omega| < \Omega_c \leq \pi$ has the DTFT frequency-response

$$H(e^{j\Omega}) = \begin{cases} 1 & \text{if } |\Omega| < \Omega_c \\ 0 & \text{if } \Omega_c < |\Omega| \leq \pi. \end{cases}$$

Thus, the inverse DTFT implies that

$$\begin{aligned}
 h[n] &= \frac{1}{2\pi} \int_{2\pi} H(e^{j\Omega}) e^{j\Omega n} d\Omega \\
 &= \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} e^{j\Omega n} d\Omega \\
 &= \frac{1}{2\pi j n} [e^{j\Omega n}]_{-\Omega_c}^{\Omega_c} \\
 &= \frac{1}{\pi n} \sin(\Omega_c n).
 \end{aligned}$$

This impulse response is not absolutely summable due to the discontinuity in $H(e^{j\Omega})$. To understand the DTFT in this case, let $H_M(e^{j\Omega})$ be the frequency response of the truncated impulse response

$$h_M[n] = \begin{cases} h[n] & \text{if } |n| \leq M \\ 0 & \text{if } |n| > M. \end{cases}$$

Due to the Gibb's phenomenon, $H_M(e^{j\Omega})$ does not converge uniformly to $H(e^{j\Omega})$. But, since $\sum_{n=-\infty}^{\infty} |h[n]|^2 < \infty$, the frequency response converges in the mean-square sense

$$\lim_{M \rightarrow \infty} \int_{-\pi}^{\pi} |H(e^{j\Omega}) - H_M(e^{j\Omega})|^2 d\Omega = 0.$$

7.3 Properties of the DTFT

The canonical properties of the DTFT are very similar to the canonical properties of the continuous-time Fourier transform (CTFS). For $x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\Omega})$ and $y[n] \xleftrightarrow{\mathcal{F}} Y(e^{j\Omega})$, we observe that:

- 1) Linearity: $ax[n] + by[n] \xleftrightarrow{\mathcal{F}} aX(e^{j\Omega}) + bY(e^{j\Omega})$
- 2) Time shift: $x[n - n_0] \xleftrightarrow{\mathcal{F}} e^{-j\Omega n_0} X(e^{j\Omega})$
- 3) Frequency shift: $e^{j\Omega_0 n} x[n] \xleftrightarrow{\mathcal{F}} X(e^{j(\Omega - \Omega_0)})$
- 4) Conjugation: $x^*[n] \xleftrightarrow{\mathcal{F}} X^*(e^{-j\Omega})$
- 5) Time flip: $x[-n] \xleftrightarrow{\mathcal{F}} X(e^{-j\Omega})$
- 6) Convolution: $x[n] * y[n] \xleftrightarrow{\mathcal{F}} X(e^{j\Omega}) Y(e^{j\Omega})$
- 7) Multiplication: $x[n] y[n] \xleftrightarrow{\mathcal{F}} X(e^{j\Omega}) \otimes Y(e^{j\Omega}) \triangleq \frac{1}{2\pi} \int_{2\pi} X(e^{j\theta}) Y(e^{j(\Omega - \theta)}) d\theta$

8) Symmetry:

$$x[n] \text{ real} \implies \operatorname{Re}\{X(e^{j\Omega})\} \text{ even, } \operatorname{Im}\{X(e^{j\Omega})\} \text{ odd, } |X(e^{j\Omega})| \text{ even, } \angle X(e^{j\Omega}) \text{ odd}$$

$$x[n] \text{ real \& even} \implies X(e^{j\Omega}) \text{ real \& even}$$

$$x[n] \text{ real \& odd} \implies X(e^{j\Omega}) \text{ pure imaginary \& odd}$$

9) Parseval's relation:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\Omega})|^2 d\Omega$$

7.4 Frequency-Domain Characterization of DT LTI Systems

The time-domain characterization of DT LTI systems is given by

$$y[n] = x[n] * h[n],$$

where $h[n]$ is the unit impulse response of the system. Taking the DTFT of this equation gives the frequency-domain characterization of DT LTI systems,

$$Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}).$$

The DTFT, $H(e^{j\Omega})$, of $h[n]$ is called the frequency response of the system.

Problem 10 (DTSP-2 2.47). A linear time-invariant system is described by the input-output relation

$$y(n) = x(n) + 2x(n-1) + x(n-2).$$

- Determine $h(n)$, the impulse response of the system.
 - Is this a stable system?
 - Determine $H(e^{j\omega})$, the frequency response of the system. Use trigonometric identities to obtain a simple expression for $H(e^{j\omega})$.
 - Plot the magnitude and phase of the frequency response.
 - Now consider a new system whose frequency response is $H_1(e^{j\omega}) = H(e^{j(\omega+\pi)})$. Determine $h_1(n)$, the impulse response of the new system.
-

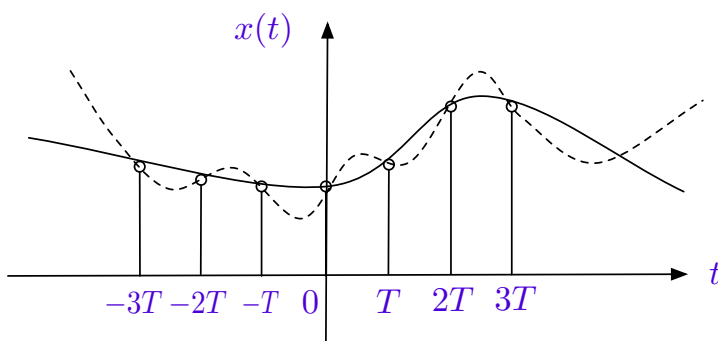
8 From Continuous Time to Discrete Time

Most signals in the physical world, by nature, are continuous-time signals $x(t)$. However, our most powerful computational tool, computer, can only process discrete-time signals $x[n]$.

Sampling: To take snapshots of $x(t)$ every T seconds to obtain the discrete-time samples

$$x[n] = x(nT), \quad n \text{ integer}$$

where T is called the sampling period. By sampling, we potentially throw out lots of information – all values of $x(t)$ between the sampling points are lost. Without any restrictions, the CT signal $x(t)$ cannot be reconstructed from its discrete-time samples $x(nT)$.



Key question: Under what conditions can we reconstruct the original CT signal $x(t)$ from its discrete-time samples $x(nT)$?

8.1 The Sampling Theorem

Definition 18. A continuous-time signal $x(t)$ is *band-limited* to ω_M if its continuous-time Fourier transform satisfies

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = 0, \quad \forall |\omega| > \omega_M.$$

Theorem 19 (Sampling Theorem). *If $x(t)$ is band-limited to ω_M , then $x(t)$ is uniquely determined by its samples $x(nT)$ if the sampling rate*

$$\omega_s = \frac{2\pi}{T} > 2\omega_M.$$

Under the context of sampling, the quantity $2\omega_M$ is usually known as the Nyquist rate.

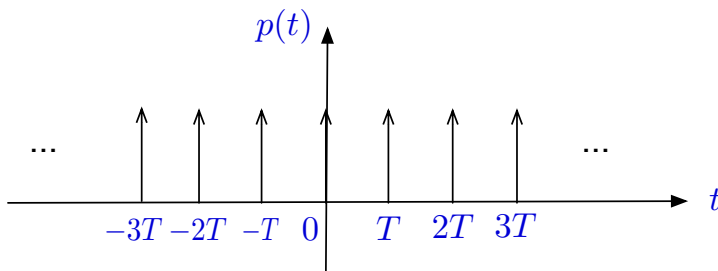
Proof of the sampling theorem: Consider the signal

$$x_p(t) = x(t)p(t)$$

where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

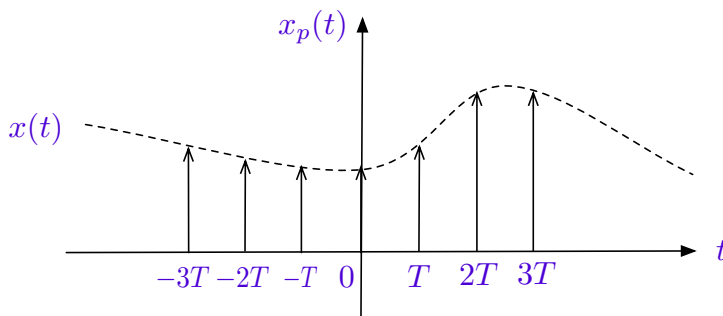
is a periodic impulse train of fundamental period T :



Then,

$$\begin{aligned} x_p(t) &= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x(t) \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT). \end{aligned}$$

Note that $x_p(t)$ only contains the information of the samples $x(nT)$.



Next, we show that if

$$X(j\omega) = 0, \quad \forall |\omega| > \omega_M$$

and

$$\omega_s = \frac{2\pi}{T} > 2\omega_M,$$

then we can perfectly reconstruct $x(t)$ from $x_p(t)$.

By the multiplication property,

$$x_p(t) = x(t)p(t) \implies X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega).$$

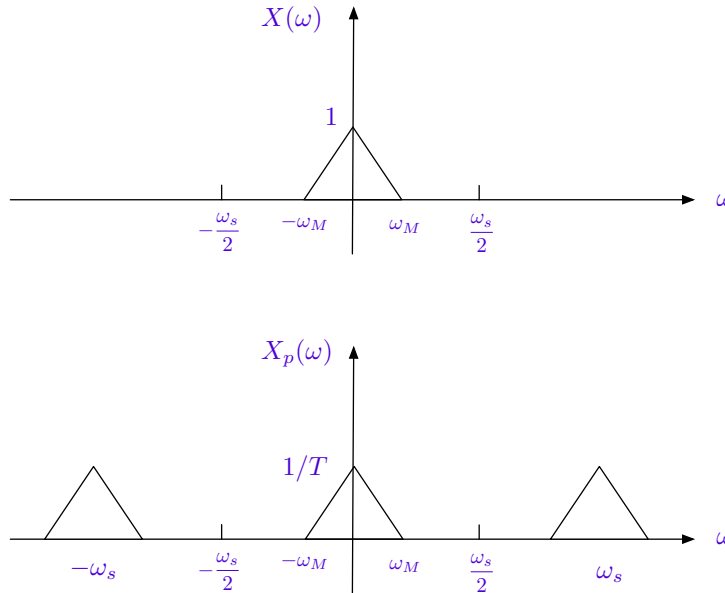
The FT of $p(t)$ is given by

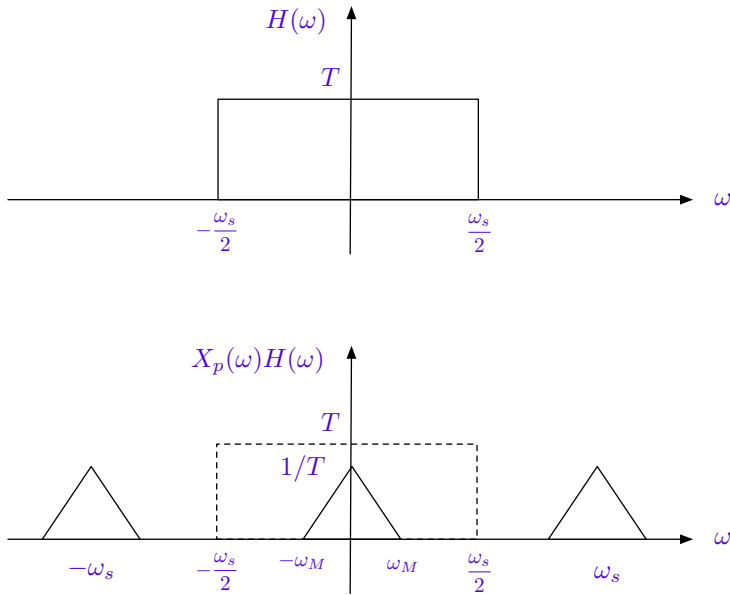
$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s).$$

Therefore,

$$\begin{aligned} X_p(j\omega) &= \frac{1}{2\pi} X(j\omega) * \left[\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \right] \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j\omega) * \delta(\omega - k\omega_s) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)). \end{aligned}$$

Key observation: Suppose that $X(j\omega) = 0$ for all $|\omega| > \omega_M$ and $\omega_s > 2\omega_M$. Then, the $X(j(\omega - k\omega_s))$ do NOT overlap for different k .





Conclusion: $X(j\omega)$ can be recovered from $X_p(j\omega)$ with an ideal low-pass filter:

$$X(j\omega) = X_p(j\omega)H(j\omega).$$

Time domain:

$$\begin{aligned} x(t) &= x_p(t) * h(t) \\ &= \left[\sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \right] * h(t) \\ &= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) * h(t) \\ &= \sum_{n=-\infty}^{\infty} x(nT)h(t - nT). \end{aligned}$$

where

$$h(t) = \frac{T \sin\left(\frac{\omega_s t}{2}\right)}{\pi t} = \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}} = \text{sinc}(t/T).$$

8.2 Undersampling and Aliasing

Under sampling: $\omega_s < 2\omega_M$, which leads to aliasing.

Aliasing: Higher frequencies of $x(t)$ “fold back” and take on the “aliases” of lower frequencies.

How can we avoid aliasing? Prefilter $x(t)$ with an ideal low-pass filter with cutoff frequency $\omega_s/2$.

