Lab 4: Physical Synthesis via the Wave Equation

Due Tuesday 4/15/19

Overview:

- This assignment should be completed with your lab partner(s).
- Each group must turn in a report composed using a word processor (e.g., Word, Pages, LaTeX, etc...) including a cover page with full names of all group members. The remaining pages should contain (in order) the answers and MATLAB scripts for the exercises. MATLAB figures can be pasted into the document or saved as PDF files. When working on the project, please follow the instructions and respond to each item listed. Your project grade is based on: (1) your MATLAB scripts, (2) your report (plots, explanations, etc. as required), and (3) your final results. For all labs, you must clearly write the problem number next to your solution and label the axes on all plots to get full credit. Submission can be done electronically in PDF format or on paper.
- Plagiarism is a very serious offense in Academia. Any figures in the paper not generated by you should labeled "Reproduced from [...]". Any portions of any simulation code (e.g., MATLAB, C, etc...) not written by you be clearly marked in your source files. The original source of any mathematical derivation or proof should be explicitly cited.

1 Overview

In this lab, we will explore the physical synthesis of a stringed instrument based on the wave equation. The wave equation models the dynamics of waves in an elastic medium (under a small amplitude approximation). The sound made by a plucked string can be synthesized by simulating the vibration of the string and the coupling of these vibrations to pressure waves in the air.

1.1 The Wave Equation

Let $u(x,t)$ denote the transverse displacement of the string (in meters) at position x (in meters) and time t (in seconds). Then, the wave equation is given by

$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2 \frac{\partial^2}{\partial x^2}u(x,t).$$

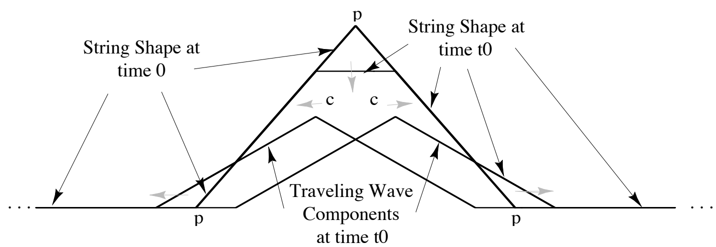
In terms of physical parameters, the propragation speed is given by $c = \sqrt{K/\rho}$ where K is the tension in the string in Newtons (i.e., kg m/s^2) and ρ is linear density of the string in kg/m. For a simple derivation of the wave equation, see `http://www.math.ubc.ca/~feldman/m256/wave.pdf`.

For the purpose of synthesis, it is useful to solve the initial value problem implied by the wave equation. In particular, the solution can be written in terms of the initial position $u(x,0)$ and the initial velocity $\frac{\partial}{\partial t}u(x,t)\big|_{t=0}$. A general form for the solution, given by d'Alembert, is

$$u(x,t) = f_r(x-ct) + f_l(x+ct),$$

where $f_r(x)$ is a right traveling waveform, $f_l(x)$ is a left traveling waveform, and $u(x,0) = f_l(x) + f_r(x)$ is the initial displacement. This solution is derived in Section 3.1. If the string is initially at rest, then setting $\frac{\partial}{\partial t}u(x,t)\big|_{t=0} = 0$ shows that one of the solutions must satisfy $f_l(x) = f_r(x) = \frac{1}{2}u(x,0)$. In this case, the equation shows that the initial shape evolves into two equal waveforms traveling in opposite directions. The figure[1] below uses d'Alembert's solution to show an initial triangular pulse evolving with time.

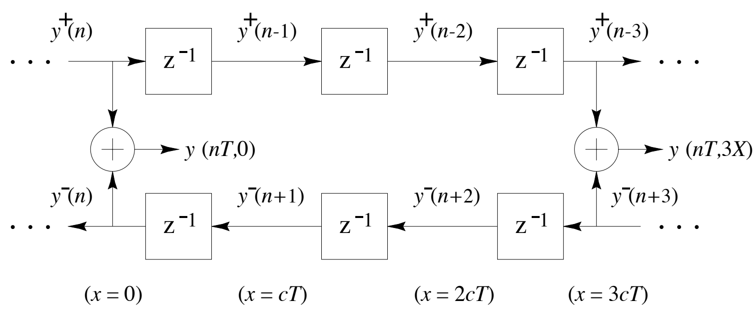
[1] All figures in this note are reproduced from [1].



We can also discretize d'Alembert's solution by sampling both space and time. If we let the sampling time be T seconds and choose the sampling interval to be $X = cT$ meters, we find that

$$
\begin{aligned}
u(mX, nT) &= f_r\left(mX - cnT\right) + f_l\left(mX + cnT\right) \\
&= f_r\left(mcT - cnT\right) + f_l\left(mcT + cnT\right) \\
&= y^+[n-m] + y^-[n+m],
\end{aligned}
$$

where $y^+[n] = f_r(-ncT)$ and $y^-[n] = f_l(ncT)$. In discrete-time, this solution has the form of an LTI system as shown below.



1.2 Rigid Termination

Now, we consider the case of a string with rigid termination at $x = 0$ and $x = L$. In this case, the wave equation should be solved using the boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$.

To understand the effect of rigid termination, we let $H(x)$ denote the Heaviside unit step function and consider a hypothetical experiment where an incident waveform $u_i(x, t) = \psi_i(x - c_1 t)H(-x)$ traveling from left to right encounters a change in medium at $x = 0$ (i.e., the propagation speed changes discontinuously). At that boundary, there is a reflected waveform $u_r(x, t) = \psi_r(x + c_1 t)H(-x)$ traveling from right to left and a transmitted waveform $u_{tr}(x, t) = \psi_{tr}(x - c_2 t)H(x)$ traveling from left to right. By assumption, the incident and reflected waveforms are zero for $x > 0$ and the transmitted waveform is zero for $x < 0$. Since the total waveform is continuous at $x = 0$ and $H(0)$, this implies that

$$
\frac{1}{2}u_i(0, t) + \frac{1}{2}u_r(0, t) = \frac{1}{2}u_{tr}(0, t).
$$

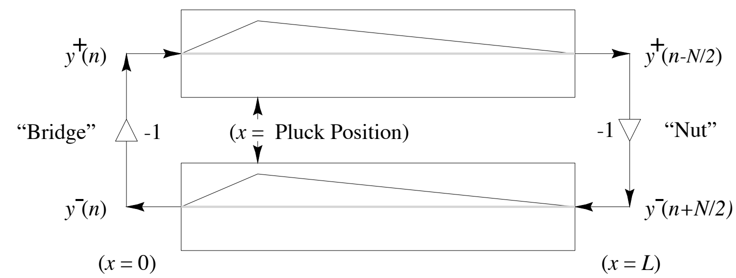
This implies that

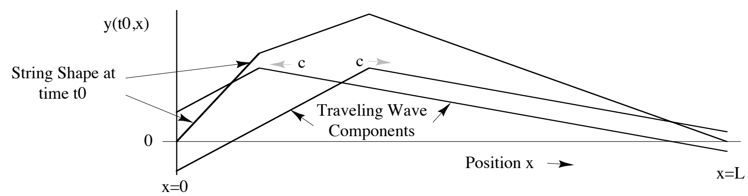
$$
\psi_i(-c_1 t) + \psi_r(c_1 t) = \psi_{tr}(-c_2 t). \tag{1}
$$

If no waveform is transmitted (i.e., $\psi_{tr}(-c_2 t) = 0$), then we find that the reflected waveform

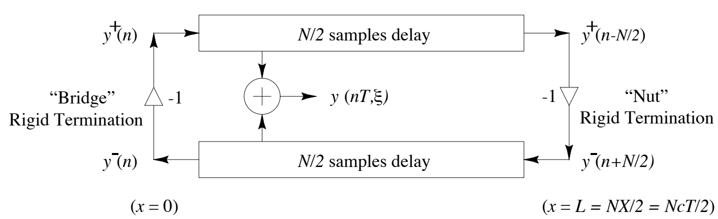
$$
\psi_r(c_1 t) = -\psi_i(-c_1 t)
$$

is negated and time reversed. This implies that traveling waveforms reflect off a rigid termination boundary with a change in sign. Illustrations of this phenomenon are shown below.





For the discretized model, this gives the following LTI system. We note that the “$N/2$ samples delay” block actually represents $N/2$ single delay elements (e.g., denoted by z^{-1} on the previous page) and only a single sample is clocked in/out of this block at each step.



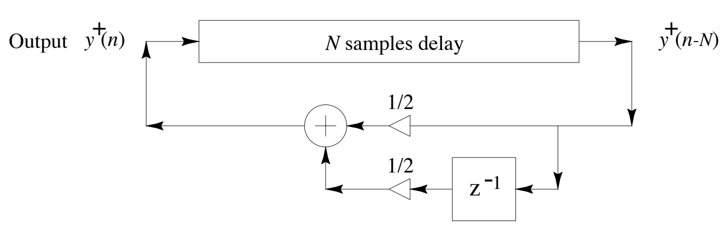
1.3 Lumped Elements

A more precise model of a plucked string would also include the filtering and energy dissipation effects in the string. If this is modeled by separate filters between each delay, then this is called *distributed filtering*. If all of the filters are commuted through the delays and implemented together, then this is called *lumped* filering. Since this system is a linear system, these two possible implementations are identical from a mathematics point of view. A common choice is to place an IIR filter at one of the rigid termination points. A reasonable choice for the filter is

$$y[n] = \gamma\left(\alpha y[n-1] + (1-\alpha)x[n]\right) \tag{2}$$

where the γ controls the decay rate and α controls the low-pass characteristic of the filter (e.g., try $\gamma = 0.99$ and $\alpha = 0.02$).

The following diagram shows a similar system except that the two $N/2$ sample delay buffers are merged into a single N sample delay buffer by adjusting the initial condition appropriately. A single FIR filter of the form $y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$ is also used to achieve a low-pass characteristic.



A more precise model of a guitar would also include string motion coupling into the guitar body, the resonance of the body, and the coupling of the body into the air. We will not model these effects precisely. One option is to design a output-stage IIR filter that approximates all of these effects.

1.4 Other Waveguide Models

For other instruments (e.g., wind and brass), it is important to consider reflected and transmitted waveforms. For completeness, the equations for this case are derived in Section 3.2. Interested students may find additional details in [1].

2 Lab Exercises

First, we will calibrate the parameters of the model. The trick is to first choose the sampling period $T = 1/F_s$ with $F_s = 44100$ Hz. By construction, the wave speed of the discrete model is 1 sampling interval (i.e., X meters) per sample time (i.e., T seconds). If we choose the length of the string to $L = NX/2$, then the resonant frequency of the string is $f = F_s/N$ Hz. Therefore, one can adjust the frequency of the synthesizer by changing parameter N. To implement the model, we will use the block diagram in Section 1.2 with two delay lines of length $N/2$. Use shift registers (i.e. 1D arrays) to represent the waves (i.e. the string's height profile) sampled in space, at one time instant. Update the registers for every clock cycle to simulate wave motion.

1. Implement the two coupled delay lines of length-$N/2$ as shown in Section 1.2, i.e. initialize all necessary parameters required for the simulation. Use $F_s = 44100$ and choose the integer value of $N/2$ which best approximates the "A" string on a guitar (i.e., 110 Hz).
2. Initialize the top shift register with a small test waveform of your choice and plot the top/bottom shift register values as a function of time. For example, try plotting the values after each time step and add a `pause(0.01)` command after the plot command so that the progression looks like a movie. Verify that the result looks as you would expect.
3. Initialize your shift registers based on setting the initial string height profile to the piecewise linear waveform shown in Section 1.2, assuming the plucked point is 1/5 of the length of the string (i.e., at sample location $\lceil N/10 \rceil$). Run the simulation and generate 2 seconds of sound by sampling the string displacement at fixed location in space (e.g., you need to sum together two entries in the shift register at the desired position on the string). Play the sound and describe it. Note that the sampled location for sound need not be the same as the initial plucked location.
4. Now add a general one-tap IIR filter to the right end of the top delay line. Use the filter suggested in (2). Generate 4 seconds of sound and play it. How does it sound?
5. Experiment with different filters and pluck points. What choices sound best to you?
6. Combine this synthesizer with the sequencer you developed for ECE 280 and generate a wave audio file containing your ECE 280 song played on a synthesized guitar.

3 Wave Equation Details

3.1 The Solution by d'Alembert

To derive d'Alembert's solution, we first use a change of variables to represent $u(x,t)$ by $v(w,z)$ satisfying

$$v(\underbrace{x-ct}_{w}, \underbrace{x+ct}_{z}) = u(x,t).$$

Using subscripts to denote partial derivatives (e.g., $u_x(x,t) \triangleq \frac{\partial}{\partial x} u(x,t)$, $u_t(x,t) \triangleq \frac{\partial}{\partial t} u(x,t)$, $u_{xx}(x,t) \triangleq \frac{\partial^2}{\partial x^2} u(x,t)$, $u_{xt}(x,t) \triangleq \frac{\partial^2}{\partial x \partial t} u(x,t)$, etc.), we can write the wave equation as $u_{tt}(x,t) = c^2 u_{xx}(x,t)$. Then, using the change of variables, we find that

$$\begin{aligned}
u_t(x,t) &= c v_z(x-ct, x+ct) - c v_w(x-ct, x+ct) \\
u_{tt}(x,t) &= c^2 \left(v_{zz}(x-ct, x+ct) - 2 v_{wz}(x-ct, x+ct) + v_{ww}(x-ct, x+ct) \right) \\
u_x(x,t) &= v_w(x-ct, x+ct) - v_z(x-ct, x+ct) \\
u_{xx}(x,t) &= v_{zz}(x-ct, x+ct) + 2 v_{wz}(x-ct, x+ct) + v_{ww}(x-ct, x+ct).
\end{aligned}$$

From this, the wave equation reduces to $v_{wz}(x-ct,x+ct) = -v_{wz}(x-ct,x+ct)$ for all $x,t \in \mathbb{R}$. This implies that

$$v_{wz}(w,z) = 0 \quad \forall w,z \in \mathbb{R}.$$

Since $\frac{\partial^2}{\partial w \partial z} = \frac{\partial}{\partial z}\frac{\partial}{\partial w}$, it follows that

$$\frac{\partial}{\partial z}\left(\frac{\partial}{\partial w} v(w,z)\right) = 0.$$

and this implies that $\frac{\partial}{\partial w} v(w,z)$ is independent of z. Defining $f_r'(w) \triangleq \frac{\partial}{\partial w} v(w,z)$, we can integrate over w for each $z \in \mathbb{R}$ to get

$$v(w,z) = f_r(w) + C(z),$$

where $C(z)$ is the constant of integration as a function of z. Finally, we define $f_l(z) = C(z)$ and observe that the solution must satisfy

$$u(x,t) = v(x-ct,x+ct) = f_r(x-ct) + f_l(x+ct).$$

Using this solution for the initial value problem shows that

$$\begin{aligned} u(x,0) &= f_r(x) + f_l(x) \\ u_t(x,0) &= -cf_r'(x) + cf_l'(x). \end{aligned}$$

Thus, $u_t(x,0) = 0$ combined with $\frac{\partial}{\partial x} u(x,0)$ shows that $f_r'(x) = f_l'(x)$. Since the solution $u(x,t)$ is defined only up to additive constants and linear terms (e.g., $\frac{\partial^2}{\partial x^2}(a + bx + ct + dxt) = 0$), we can always choose a solution that satisfies $f_r(x) = f_l(x) = \frac{1}{2}u(x,0)$.

3.2 The Case of Partial Transmission

When the transmission medium changes abruptly, there may be partial transmission. To analyze this, we will use the notation from Section˜1.2. If the waveform is partially transmitted, then slope continuity near $x = 0$ implies that

$$\frac{\partial}{\partial x}\left(u_i(x,t) + u_r(x,t)\right)\bigg|_{x=0^-} = \frac{\partial}{\partial x}\left(u_{tr}(x,t)\right)\bigg|_{x=0^+}.$$

Note that we avoid taking the derivative exactly at $x = 0$ due to the step function discontinuity. Plugging in the formulas, one finds that

$$\psi_i'(-c_1 t) + \psi_r'(c_1 t) = \psi_{tr}'(-c_2 t). \tag{3}$$

Taking the derivative of the continuity equation (1) shows that

$$\psi_{tr}'(-c_2 t) = \frac{-c_1}{-c_2}\psi_i'(-c_1 t) + \frac{c_1}{-c_2}\psi_r'(c_1 t).$$

Solving (1) and (3) together shows that

$$\begin{aligned} \psi_r'(c_1 t) &= \frac{c_1 - c_2}{c_1 + c_2}\psi_i'(-c_1 t) \\ \psi_{tr}'(-c_2 t) &= \frac{2c_1}{c_1 + c_2}\psi_i'(-c_1 t). \end{aligned}$$

Integrating both sides of these equations, we find that

$$\begin{aligned} \psi_r(c_1 t) &= \frac{c_2 - c_1}{c_1 + c_2}\psi_i(-c_1 t) \\ \psi_{tr}(-c_2 t) &= \frac{2c_2}{c_1 + c_2}\psi_i(-c_1 t). \end{aligned}$$

References

[1] J. O. Smith, "Physical modeling using digital waveguides," *Computer Music Journal*, vol. 16, no. 4, pp. 74–91, 1992.