Lab 5: Spatial Signal Processing

Due Monday 04/29/19

Overview:

- This assignment should be completed with your chosen lab partner(s).

- Each group must turn in a report composed using a word processor (e.g., Word, Pages, LATEX, etc...) including a cover page with full names of all group members. The remaining pages should contain (in order) the answers and MATLAB scripts for the exercises. MATLAB figures can be pasted into the document or saved as PDF files. When working on the project, please follow the instructions and respond to each item listed. Your project grade is based on: (1) your MATLAB scripts, (2) your report (plots, explanations, etc. as required), and (3) your final results. For all labs, you must clearly write the problem number next to your solution and label the axes on all plots to get full credit. Submission can be done electronically in PDF format or on paper.

- Plagiarism is a very serious offense in Academia. Any figures in the paper not generated by you should labeled "Reproduced from [...]". Any portions of any simulation code (e.g., MATLAB, C, etc...) not written by you be clearly marked in your source files. The original source of any mathematical derivation or proof should be explicitly cited.

1 Overview

It is often possible to measure a signal of interest at different spatial locations. Once measured, these multiple signals can be used to improve system performance. For example, modern cell phones and wireless routers typically contain multiple antennas (and receive chains) that sample electromagnetic radiation at multiple points in space. These additional signals are used to provide robustness against interference and multipath fading. Many vehicles also contain microphone arrays for hands-free operation of cellular phones. These arrays improve performance by minimizing the effects of road noise. The underlying principles behind spatial signal processing are largely independent of the physical medium and, thus, the ideas in this handout can applied to many different systems.

2 System Model

2.1 Propagation Model

An isotropic point source generates an amplitude wave that propagates uniformly in all directions. A good analogy is dropping a small pebble in the middle of a very large lake. The result is a circular ripple that propagates at the same speed in all directions. The response observed at other points in the lake is delayed by an amount of time proportional to the distance from the initial disturbance. Moreover, if multiple pebbles are dropped at many different points in the lake, then the overall response will be given by the sum of the individual responses.

Consider a spatial medium where the speed of propagation is c (in meters per second) and suppose a point source located at $\underline{x}_1 \in \mathbb{R}^3$ generates the signal $s_1(t)$. Then, the response at an arbitrary location $\underline{x} \in \mathbb{R}^3$ is assumed to be

$$s_1(\underline{x}, t) \triangleq \frac{1}{\|\underline{x} - \underline{x}_1\|} s_1 \left(t - \frac{1}{c} \|\underline{x} - \underline{x}_1\| \right), \tag{1}$$

where $\|\underline{x} - \underline{x}_1\|$ is the Euclidean distance between $\underline{x}$ and $\underline{x}_1$. We note that the decay rate of amplitude with distance follows from assuming lossless wave propagation and conservation of energy. If there are multiple point sources $s_1(t), \ldots, s_N(t)$ located at points $\underline{x}_1, \ldots, \underline{x}_N \in \mathbb{R}^3$, then the overall response at $\underline{x}$ is given by

$$u(\underline{x}, t) \triangleq \sum_{n=1}^{N} s_n(\underline{x}, t) = \sum_{n=1}^{N} \frac{1}{\|\underline{x} - \underline{x}_n\|} s_n \left(t - \frac{1}{c} \|\underline{x} - \underline{x}_n\| \right).$$

2.2 Spatial Filter

Now, consider the case where there are m sensors located at positions $\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_m$. In this case, the response at sensor m of source n is given by $a_{n,m}s_n(t - \tau_{n,m})$, where $a_{n,m}$ is amplitude and $\tau_{n,m}$ is the delay (cf., (1)). Thus, the overall response at sensor m is given by

$$u_m(t) \triangleq \sum_{n=1}^{N} a_{n,m}s_n\left(t - \tau_{n,m}\right).$$

To work with this model in Matlab, we will also switch to discrete time (with sample period $T = 1/F_s$) and assume that each received signal $u_m(t)$ is band-limited to $F_s/2$ Hz before sampling (e.g., perhaps by an ideal low-pass filter during reception) to get

$$u_m[\ell] \triangleq u_m\left(\ell T\right).$$

In this case, one might as well assume that each source signal $s_n(t)$ was also band-limited to $F_s/2$ Hz before sampling.

The spatial filter operates by filtering each of the observed signals with different filters and then summing the outputs together. Specifically, the output of the spatial filter is defined to be the linear combination of the observed signals through filters $h_1[\ell], \ldots, h_M[\ell]$ (supported[1] on $\ell \in \{\tau_1, \tau_1 + 1, \ldots, \tau_2\}$) defined by

$$v[\ell] = \sum_{m=1}^{M} \sum_{\tau=\tau_1}^{\tau_2} h_m[\tau]u_m[\ell - \tau].$$

Since $v[\ell]$ is a linear function of $h_m[\tau]$ for fixed $u_m[\ell]$, we can restrict our attention to[2] $\ell \in \{\ell_1, \ell_1 + 1, \ldots, \ell_2\}$ and write this as the matrix equation

$$\underline{v} = U\underline{h},$$

where the vectors $\underline{v}$ and $\underline{h}$ are given by

$$\underline{v} = [v[\ell_1], v[\ell_1 + 1], \ldots, v[\ell_2]]^T$$
$$\underline{h} = [h_1[\tau_1], h_1[\tau_1 + 1], \ldots, h_1[\tau_2], h_2[\tau_1], \ldots, h_2[\tau_2], h_3[\tau_1], \ldots, h_M[\tau_2]]^T$$

and the matrix U is given by

$$U = \begin{bmatrix} u_1[\ell_1 - \tau_1] & u_1[\ell_1 - \tau_1 - 1] & \cdots & u_1[\ell_1 - \tau_2] & u_2[\ell_1 - \tau_1] & \cdots & u_M[\ell_1 - \tau_2] \\ u_1[\ell_1 + 1 - \tau_1] & u_1[\ell_1 + 1 - \tau_1 - 1] & \cdots & u_1[\ell_1 + 1 - \tau_2] & u_2[\ell_1 + 1 - \tau_1] & \cdots & u_M[\ell_1 + 1 - \tau_2] \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ u_1[\ell_2 - \tau_1] & u_1[\ell_2 - \tau_1 - 1] & \cdots & u_1[\ell_2 - \tau_2] & u_2[\ell_2 - \tau_1] & \cdots & u_M[\ell_2 - \tau_2] \end{bmatrix}.$$

2.3 Designing the Spatial Filter with a Known Training Signal

The primary question becomes how should one choose the values of the combining filters $h_m[\tau]$ (or equivalently $\underline{h}$) in order to best reconstruct $s_1[\ell]$. In most real systems, this is difficult because one cannot assume knowledge of the desired signal $s_1[\ell]$. But, if the system includes a training phase based on a known signal, then one can solve for the filters during training using least squares.

In its simplest form, using least squares to design the spatial filter involves minimizing the power in the error signal

$$e[\ell] = v[\ell] - s_1[\ell].$$

In vector notation, we define

$$\underline{s} = [s_1[\ell_1], s_1[\ell_1 + 1], \ldots, s_1[\ell_2]]^T$$
$$\underline{e} = [e[\ell_1], e[\ell_1 + 1], \ldots, e[\ell_2]]^T$$

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[1] The constants τ_1, τ_2 can be chosen based on the maximum delay experienced by any signal and the frequency content of the sources.

[2] The constants ℓ_1, ℓ_2 can be chosen based on the observation length of the signals.

so that $\underline{e} = U\underline{h} - \underline{s}$. Then, minimizing the power in the error signal is equivalent to solving the linear least-squares minimization problem given by

$$\min_{\underline{h} \in \mathbb{R}^{M(\tau_2 - \tau_1 + 1)}} \|U\underline{h} - \underline{s}\|^2 .$$

In general, the matrix U should be full rank as long as $\ell_2 - \ell_1 + 1 \gg M(\tau_2 - \tau_1 + 1)$ and there is some noise in the system. If U is full-rank, then the unique solution is given by

$$\hat{\underline{h}} = (U^T U)^{-1} U^T \underline{s}.$$

Solutions computed using this method automatically adjust the spatial-frequency response to optimize the SNR of the desired signal. That means that both the spectral and spatial characteristics of the signal are exploited to increase the SNR.

2.4   Warmup Exercise

Suppose you take K real measurements $y_1, y_2, \ldots, y_K$ of an apparatus with control settings $x_1, x_2, \ldots, x_K$. For this particular apparatus, a linear response $y = ax + b$ is expected. The exercise is to use linear least squares to solve the overdetermined set of equations $y_i = ax_i + b$ for a, b with $K = 5$ and

i	1	2	3	4	5
x_i	1	2	4	5	7
y_i	3	3	-1	-5	-8

(a) Write down the 5×2 matrix U such that these equations are given by

$$U \begin{bmatrix} a \\ b \end{bmatrix} = \underline{y}.$$

(b) Compute the least squares solution $[\hat{a} \ \hat{b}] = (U^T U)^{-1} U^T \underline{y}$.

(c) Plot estimated line $y = \hat{a}x + \hat{b}$ for $x = 1, 2, \ldots, 7$ along with the measured data.

2.5   Spatial Filter Exercise

In this exercise, you are given the desired signal (target.wav) and simulated recordings of noisy signals recorded at 4 microphones (fmic_1_4.wav, mic_2_4.wav, ..., mic_4_4.wav). Your task is to evaluate the performance of spatial filtering systems with different reconstruction strategies and different numbers of microphones.

It is worth noting that the delays of the desired signal (in samples) for the 4 simulated recordings are $(\tau_{1,1}, \tau_{1,2}, \tau_{1,3}, \tau_{1,4}) = (682, 621, 573, 543)$, respectively. In principle, one could estimate these delays from the cross-correlation $r_{u_m s_1}[\tau] = \sum_\ell u_m[\ell] s_1[\ell - \tau]$ because this function should have a peak around $\tau = \tau_{1,m}$.

(a) Estimate the desired signal by compensating for the delay and averaging from 2 to 4 microphones. In particular, for $M = 2, 3, 4$, compute

$$y[\ell] = \sum_{m=1}^{M} u_m[\ell + \tau_{1,m}]$$

and judge its sound quality.

(b) For $M = 2, 3, 4$, setup and solve the spatial filter design problem described in Section 2.3 with $\tau_1 = -717$ and $\tau_2 = -517$. Describe the sound quality of each reconstruction and plot the magnitude frequency responses of the filters in each case (e.g., use [H,w]=freqz(h,1) for each filter). Hint: To reduce indexing hassles, you can generate the filtered output signal by computing $\underline{v} = U\hat{\underline{h}}$ instead of extracting and implementing the filters.

(c) Save the $M = 3$ reconstruction from part (b) as a wave file and submit this with your lab.