Mathematical Review for Signal and Systems

1 Trigonometric Identities

It will be useful to memorize $\sin\theta, \cos\theta, \tan\theta$ values for $\theta = 0, \pi/3, \pi/4, \pi/2$ and $\pi \pm \theta$, $2\pi - \theta$ for the above values of θ.

The following identities involving sine and cosine functions will be useful

$$\begin{aligned}
\sin(\theta \pm \phi) &= \sin\theta \ \cos\phi \pm \cos\theta \ \sin\phi \\
\cos(\theta \pm \phi) &= \cos\theta \ \cos\phi \mp \sin\theta \ \sin\phi \\
\sin\theta \sin\phi &= \frac{1}{2}\left[\cos(\theta - \phi) - \cos(\theta + \phi)\right] \\
\cos\theta \cos\phi &= \frac{1}{2}\left[\cos(\theta - \phi) + \cos(\theta + \phi)\right] \\
\sin\theta \cos\phi &= \frac{1}{2}\left[\sin(\theta - \phi) + \sin(\theta + \phi)\right] \qquad (1)
\end{aligned}$$

The following special case of the above formulas are also very useful to commit to memory

$$\begin{aligned}
\cos^2\theta &= \frac{1}{2}(1 + \cos 2\theta) \\
\sin^2\theta &= \frac{1}{2}(1 - \cos 2\theta) \\
\cos(2\theta) &= 2\cos^2\theta - 1 = 1 - 2\sin^2\theta
\end{aligned}$$

Any two numbers a and b can be written as $a = r\cos\theta$ and $b = r\sin\theta$, where $r = \sqrt{a^2 + b^2}$ and $\tan(\theta) = b/a$. If $a > 0$, then one also has $\theta = \tan^{-1}\left(\frac{b}{a}\right)$, where $\tan^{-1}(x)$ denotes the principal branch of the inverse tangent which satisfies $\tan^{-1}(\tan(x)) = x$ for $-\pi/2 < x < \pi/2$.

The cosine, sine and exponential functions have infinite series (Maclaurin's series) expansions given by

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \ldots \qquad (2)$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \ldots \qquad (3)$$

$$e^\theta = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} + \ldots \qquad (4)$$

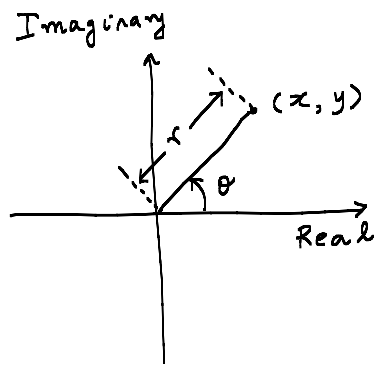
where θ is in radians.

2 Complex Numbers

We will use the letter j to refer to the imaginary number $\sqrt{-1}$. Even though j is not a real number, we can perform all arithmetic operations such as addition, subtraction, multiplication, division with j using the algebra of real numbers.

2.1 Cartesian Form

A complex number z is any number of the form $z = x + jy$, where x is called the real part of z and y is called the imaginary part of z. Note: The imaginary part is not jy, rather it is only y. It is important to stick to this terminology, otherwise computations can go wrong. Often, it is useful to think of a complex number $z = x + jy$ as a vector in a two-dimensional plane as shown in the figure below, where x is the X-coordinate and y is Y-coordinate of the vector. Due to this relationship between a complex number and the corresponding vector, we will abuse the terminology and use the terms complex number and vector interchangeably, if the context should resolve any possible ambiguity. For example, a complex number is said to lie in the first quadrant (or, second quadrant etc) if the corresponding vector lies in the first quadrant (or, second quadrant etc).



2.2 Polar or Exponential Form, Magnitude and Phase

When a complex number is thought of as a vector in two dimensions, the X coordinate x and the Y coordinate y can be expressed in terms of the length of the vector r and the angle made by this vector with the positive X-axis, namely θ. Since $x = r\cos\theta$ and $y = r\sin\theta$, z can be expressed as

$$z = r\cos\theta + jr\sin\theta, \tag{5}$$

where θ can be in degrees or radians (usually radians) and recall that 2π rad $= 360\,^\circ$. r is called the magnitude of z, denoted by $|z|$ and θ is called the phase of the complex number z, denoted by $\arg z$ or $\angle z$.

Using Euler's identities z can be written as

$$z = r\cos\theta + jr\sin\theta = re^{j\theta} \tag{6}$$

This is known as the polar form or exponential form and it is very important to be able to convert a complex number from cartesian form to exponential form and vice versa. It is easy to see that x, y, r and θ are related according to

$$\begin{aligned} x = r\cos\theta, &\qquad y = r\sin\theta \\ r = \sqrt{x^2+y^2}, &\qquad \tan(\theta) = \frac{y}{x}. \end{aligned} \tag{7}$$

The value of θ (for all x, y) is given by

$$\theta = \begin{cases} \arctan(\frac{y}{x}) & x > 0 \\ \pi + \arctan(\frac{y}{x}) & y \geq 0, x < 0 \\ -\pi + \arctan(\frac{y}{x}) & y < 0, x < 0 \\ \frac{\pi}{2} & y > 0, x = 0 \\ -\frac{\pi}{2} & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0. \end{cases}$$

Example 1. *It is very useful to know the polar form for often used complex numbers such as* $1, j, -j, -1$. *They are given by* $1 = e^{j0}, -1 = e^{j\pi}, j = e^{j\frac{\pi}{2}}, -j = e^{-j\frac{\pi}{2}}$.

Caution: r must be positive in the above expression. For example, if $z = -2e^{j\frac{\pi}{4}}$, we must rewrite z as $z = 2e^{j\frac{5\pi}{4}}$ and interpret r as 2 instead of -2.

Caution: The expression for θ in (7) does not identify θ uniquely, since $\tan(\theta) = \frac{y}{x}$ also implies that $\tan(\theta \pm \pi) = \frac{y}{x}$. However, one can use the signs of (x, y) to determine which quadrant contains this vector and then adjust the value of $\theta = \tan^{-1}(y/x)$ accordingly. This is done automatically by the by the more complicated expression for θ.

Example 2. *Suppose* $z_1 = \frac{\sqrt{3}}{2} + j\frac{1}{2}$ *and* $z_2 = -\frac{\sqrt{3}}{2} - j\frac{1}{2}$. *It is easy to see that* $\tan^{-1}\left(\frac{y_1}{x_1}\right) = \tan^{-1}\left(\frac{y_2}{x_2}\right)$. *However,* z_1 *is complex number in the first quadrant, whereas* z_2 *is a complex number is the 3rd quadrant. Therefore,* θ_1 *should be* $\pi/6$ *and* θ_2 *should be* $7\pi/6$.

One important aspect of the polar form for a complex number is that adding 2π to the angle does not change the complex number. Particularly, $\boxed{re^{j\theta} = re^{(j\theta + 2k\pi)}}$. While this seems innocuous at first, this fact will be repeatedly used in the course. An immediate example of where this is useful is given in Section 2.6.

Example 3. *Express* $e^{j2\pi}, e^{-j\pi}, e^{j\frac{3\pi}{2}}, e^{j\frac{9\pi}{2}}$ *in Cartesian form.*

2.3 Euler's Identities

In (4) if one replaces θ by $j\theta$ and $-j\theta$, we get the following two equations, respectively.

$$e^{j\theta} = 1 + \frac{j\theta}{1!} + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \dots \tag{8}$$

$$e^{-j\theta} = 1 - \frac{j\theta}{1!} + \frac{(j\theta)^2}{2!} - \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} - \frac{(j\theta)^5}{5!} + \dots \tag{9}$$

From (8), (9), (2) and (3) the following relationship can be seen to be true

$$\begin{aligned} e^{j\theta} &= \cos\theta + j\sin\theta \\ e^{-j\theta} &= \cos\theta - j\sin\theta \\ \cos\theta &= \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \\ \sin\theta &= \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) \end{aligned}$$

2.4 Conjugate

The conjugate of a complex number $z = x + jy$ is given by $z^* = x - jy$. When z is written in polar form as $z = re^{j\theta}$, the complex conjugate is given by $z^* = re^{-j\theta}$. In general, to compute the conjugate of a complex number, replace j by $-j$ everywhere.

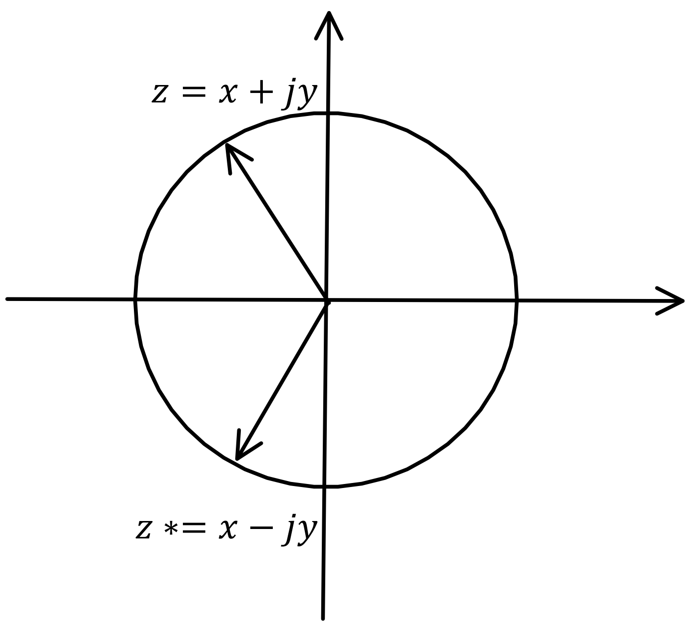


Figure 1: Complex conjugate

2.5 Operations on two complex numbers

Let $z = x + jy = re^{j\theta}$, $z_1 = x_1 + jy_1 = r_1 e^{j\theta_1}$ and $z_2 = x_2 + jy_2 = r_2 e^{j\theta_2}$

$$
\begin{aligned}
z_1 \pm z_2 &= (x_1 + x_2) + j(y_1 \pm y_2) && (10)\\
z_1 z_2 &= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1) = r_1 r_2 e^{j(\theta_1+\theta_2)} && (11)\\
zz^* &= x^2 + y^2 = r^2\\
|z| &= \sqrt{zz^*} = r\\
\frac{z_1}{z_2} &= \frac{(x_1 + jy_1)}{(x_2 + jy_2)} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}
\end{aligned}
$$

Based on the above operations, the following facts about complex number can be verified.

$$
\begin{aligned}
(z_1 + z_2)^* &= z_1^* + z_2^*\\
(z_1 \; z_2)^* &= z_1^* \; z_2^*\\
\left(\frac{z_1}{z_2}\right)^* &= \frac{z_1^*}{z_2^*}\\
|z_1 - z_2| &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\\
|z_1 z_2| &= |z_1||z_2| = r_1 r_2\\
\left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|} = \frac{r_1}{r_2}
\end{aligned}
$$

2.6 n^{th} power and n^{th} roots of a complex number

Let $z_0 = x_0 + jy_0 = r_0 e^{j\theta_0}$. For any integer n, the nth power of z, z^n is simply obtained by using (11) n times. In the polar form, $z_0^n = r_0^n e^{jn\theta_0}$. Just like how the two real numbers 1 and -1 have the same square, different complex numbers can have the same nth power.

Consider the set of distinct complex numbers $z_k = e^{j\theta_0 + \frac{2\pi k}{n}}$. All the z_ks are different have the same nth power for $k = 0, 1, 2, \ldots, n-1$. We can see this by raising z_k to the nth power to get

$$
z_k^n = \left(e^{j\theta_0 + \frac{2\pi k}{n}}\right)^n = e^{jn\theta_0 + 2\pi k} = e^{jn\theta_0}. \qquad (12)
$$

The nth root of z is a bit more interesting and tricky. Any complex number z which is the solution to the nth degree equation

$$
z^n - z_0 = 0
$$

is an nth root of z_0. The fundamental theorem of algebra states that an nth degree equation has exactly n (possibly complex) roots. Hence, every complex number z_0 has exactly n, nth roots. These roots can be found as follows (notice that the use of the fact that $e^{j\theta} = e^{j(\theta+2\pi k)}$ is key.

$$
\begin{aligned}
z^n = z_0 \;\; &\Rightarrow \;\; r^n e^{jn\theta} = r_0 e^{j\theta_0} = r_0 e^{j(\theta_0 + 2k\pi)}\\
&\Rightarrow \;\; r = \sqrt[n]{r_0}, \quad , \theta = \frac{\theta_0 + 2k\pi}{n} \text{ for } k = 0, 1, 2, \ldots n-1
\end{aligned}
$$

Clearly, computing nth roots is much easier in the polar form than in the cartesian form.

Example 4. *Find the third roots of unity* $\sqrt[3]{1}$

Since $1 = 1e^{j0}$*, this corresponds to* $r_0 = 1, \theta_0 = 0$*. Hence, the three roots of unity are given by*

$$r = 1, \quad \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}.$$

In cartesian coordinates, they are $(1+j0)$, $\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$*. These are referred to as* $1, \omega, \omega^2$ *sometimes. The three roots are shown in Figure 2.*

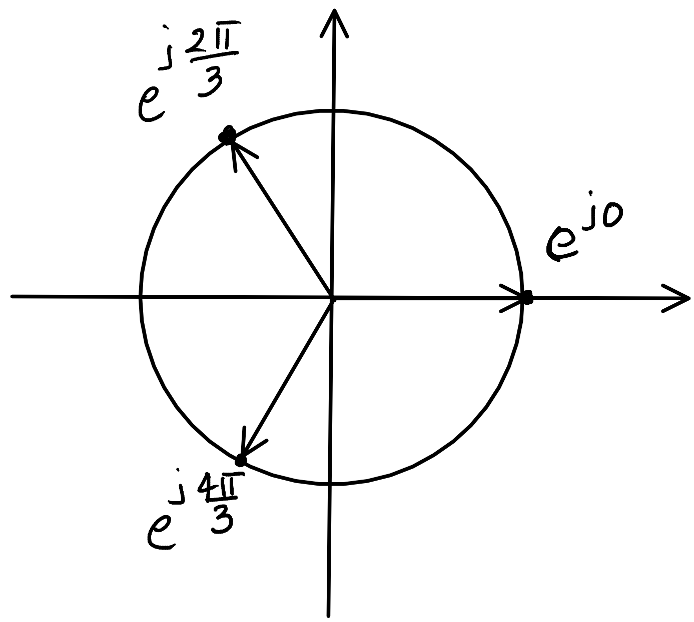


Figure 2: Cube roots of unity

2.7 Functions of a complex variable

Let $f(z)$ be a complex function of a complex variable z, i.e., for every z, $f(z)$ is a complex number. Note that a real number is also considered as a complex number and, hence, $f(z)$ could have a zero imaginary part. Examples of functions include $f(z) = |z|, f(z) = \arg(z), f(z) = z^n, f(z) = \exp(z), f(z) = \log(z)$, etc. Both these exponential and logarithmic functions can be interpreted using Euler's identity as follows.

$$f(z) = \exp(z) = e^x e^{jy} = e^x \cos y + j e^x \sin y$$

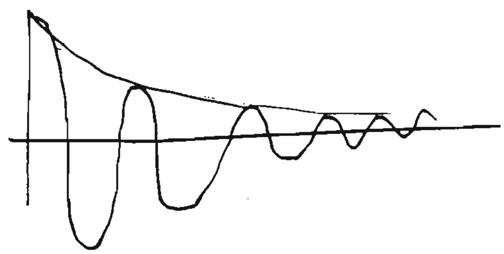


Figure 3: $\Re\{\exp z\}$ vs $\Re\{z\}$

The real part of $f(z)$ is plotted as a function of the real part of z, namely x for the case $x < 0$ in Fig. 3.

The logarithm function can be also interpreted using Euler's identity as $\log(z) = \log\left(re^{j\theta}\right) = \log r + j\theta$.

2.8 Complex functions of a real variable

You may be used to dealing with functions of a variable such as $y = f(x)$, where x is called the independent variable and y is called the dependent variable and typically, y takes real values when x takes real values. In this course, we will be interested in complex functions of a real variable such as time or frequency. Such a function, normally denoted as $x(t)$ or $X(\omega)$ is a function which takes a complex value for every real value of the independent variable t or ω. Pay attention to the notation carefully - t or ω now becomes the independent variable and $x(t)$ or $X(\omega)$ now becomes the dependent variable. We can also think of the complex function as the combination of two real functions of the independent variable, one for the real part of $x(t)$ and one for the imaginary part of $x(t)$.

When dealing with real functions of a real variable, you may be used to plotting the function $x(t)$ as a function of t. However, when $x(t)$ is a complex function, there is a problem in plotting this function since for every value of t, we need to plot a complex number. In this case, we do one of two things - either we plot the real part of $x(t)$ versus t and plot the imaginary part of $x(t)$ versus t, or we plot $|x(t)|$ versus t and $\arg(x(t))$ versus t. Either of these is fine, but we do need two plots to effectively understand how $x(t)$ changes with t.

Example 5. *Consider the function $x(t) = e^{j2\pi t} = \cos 2\pi t + j \sin 2\pi t$ for all real values of t. This is clearly a complex function of a real variable t. $\Re\{x(t)\}, \Im\{x(t)\}, |x(t)|, arg(x(t))$ are all real functions of the real variable t. Hence, we can plot $\Re\{x(t)\}$ versus t and $\Im\{x(t)\}$ versus t or we can plot $|x(t)$ versus t and $arg(x(t))$ versus t as shown in Fig. 4*

Example 6. *Consider the function $H(\omega) = \frac{1}{1+j\omega}$, where ω is a real variable. Roughly sketch*

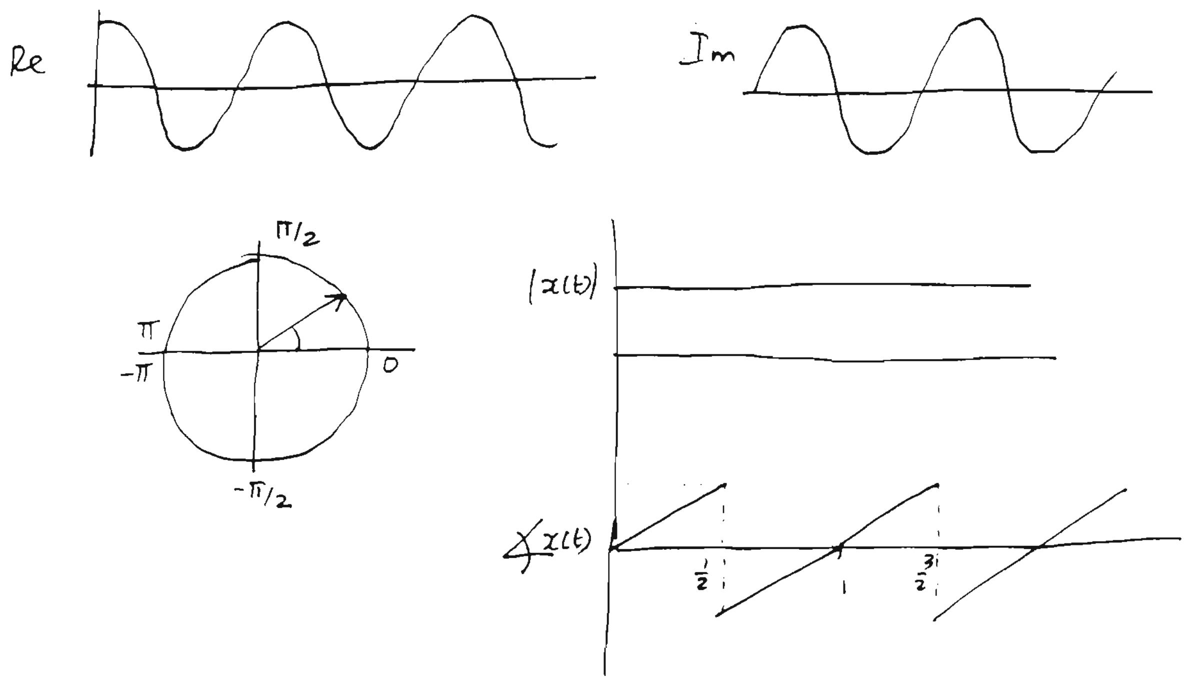


Figure 4: Plot of $\Re\{x(t)\}, \Im\{x(t)\}, |x(t)|, \angle x(t)$ versus t for $x(t) = e^{j2\pi t}$.

the magnitude and phase of $H(\omega)$ as a function of ω.

$$
\begin{aligned}
H(\omega) &= \frac{1}{1+j\omega} \\
|H(\omega)| &= \frac{1}{\sqrt{1+\omega^2}} \\
\angle(H(\omega)) &= 0 - \tan^{-1}\omega
\end{aligned}
$$

A plot of $|H(\omega)|$ versus ω and $\angle(H(\omega))$ versus ω is shown in Fig. 5.

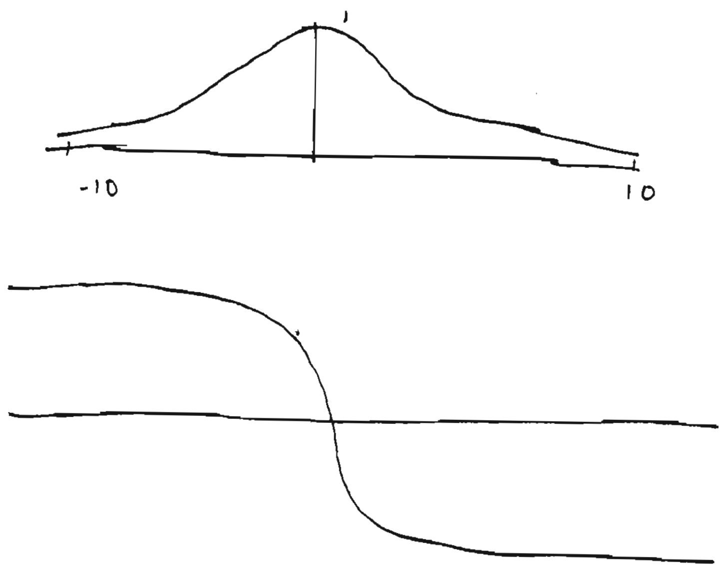


Figure 5: Plot of $H(\omega)$ vs ω and $\angle H(\omega)$ versus ω for $H(\omega) = \frac{1}{1+j\omega}$.

Example 7. *Consider the function $X(\omega) = \frac{j\omega}{1+j\omega}$, where ω is a real variable. Roughly sketch the magnitude and phase of $X(\omega)$ as a function of ω.*

$$
\begin{aligned}
X(\omega) &= \frac{j\omega}{1+j\omega} \\
|X(\omega)| &= \frac{|\omega|}{\sqrt{1+\omega^2}} \\
\angle(X(\omega)) &= \begin{cases} -\frac{\pi}{2} - \tan^{-1}\omega & ,\omega < 0 \\ \frac{\pi}{2} - \tan^{-1}\omega & ,\omega > 0 \end{cases}
\end{aligned}
$$

A plot of $|X(\omega)|$ versus ω and $\angle(X(\omega))$ versus ω is shown in Fig. 6

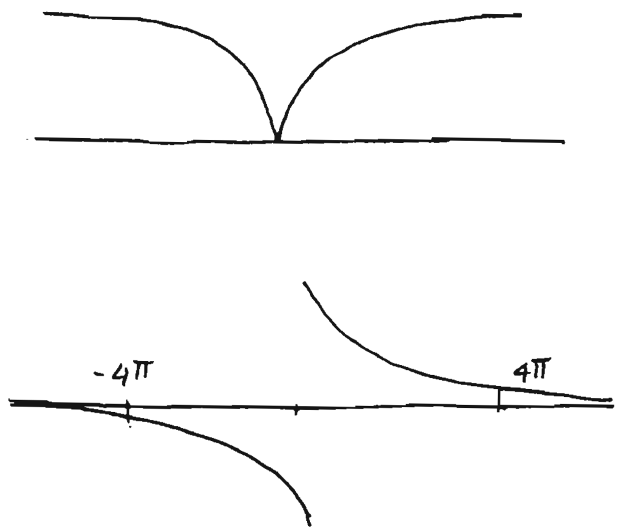


Figure 6: Plot of $X(\omega)$ vs ω and $\angle X(\omega)$ versus ω for $X(\omega) = \frac{j\omega}{1+j\omega}$.

2.9 Examples

1. Let $z_1 = 2e^{j\pi/4}$ and $z_2 = 8e^{j\pi/3}$. Find

   a) $2z_1 - z_2$

   b) $\frac{1}{z_1}$

   c) $\frac{z_1}{z_2^2}$

   d) $\sqrt[3]{z_2}$

2. What is j^j?

3. Let z be any complex number. Is it true that $(e^z)^\star = e^{z^\star}$?

4. Plot the magnitude and phase of the function $X(f) = e^{j\pi f} + e^{j3\pi f}$, for $-1 \leq f \leq 1$.

5. Prove that
$$\int e^{ax} \cos(bx)\, dx = \frac{e^{ax}}{a^2 + b^2} \left(a\cos(bx) + b\sin(bx)\right)$$

2.10 References

A good online reference for complex numbers is the wiki page `http://en.wikipedia.org/wiki/Complex_number`.

3 Geometric Series

A series of the form $ab^k, ab^{k+1}, \ldots, aa^{k+l}$, where a and b can be any *complex* number is called a geometric series with $l+1$ terms. For example, $1, \frac{1}{2}, \frac{1}{4}, \ldots$ is an infinite geometric series with $a = 1$, $b = \frac{1}{2}$. You may have seen these before, but in this class often we will be interested in the case when b (and a) are complex numbers. Luckily, nothing changes from when a and b are just real numbers.

We will particularly be interested in writing a closed form expression for the sum of consecutive terms of a geometric series. The most general result that you should *memorize* is that

$$\sum_{n=k}^{l} a\ b^n = \begin{cases} a\left(\frac{b^k - b^{l+1}}{1-b}\right), & b \neq 1; \\ a(l-k+1), & b = 1. \end{cases} \tag{13}$$

A few special cases of the above general result are important. Just convince yourself that these are true

$$\begin{aligned} \sum_{n=k}^{\infty} a\ b^n &= a\left(\frac{b^k}{1-b}\right), |b| < 1; \\ \sum_{n=-k}^{-\infty} a\ b^n &= ab^{-k}\left(\frac{b}{b-1}\right), |b| > 1; \end{aligned}$$

Another useful result is

$$\sum_{n=0}^{\infty} nb^n = \frac{b}{(1-b)^2}, \quad |b| < 1 \tag{14}$$

Here are couple of examples to try out

1. For any two given integers k and M, what is $\sum_{n=0}^{M-1} e^{\frac{j2\pi kn}{M}}$?

2. Just for intellectual curiosity - Can you prove the results in (13) and (14)?

4 Integration by Parts

Let $u(x)$ and $v(x)$ be differentiable and either $u'(x)$ or $v'(x)$ be continuous. Then,

$$\int_a^b u(x)v'(x)\mathrm{d}x = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\mathrm{d}x.$$

Example 8. *For any ω, consider the integral*

$$\int_0^1 te^{-j\omega t}\mathrm{d}t$$

and let $u(t) = t$ *and* $v'(t) = e^{-j\omega t}$*. Then,* $u'(t) = 1$ *and* $v(t) = -\frac{1}{j\omega}e^{-j\omega t}$*. Thus, integration by parts shows that*

$$\begin{aligned}
\int_0^1 te^{-j\omega t}\mathrm{d}t &= -\frac{t}{j\omega}e^{-j\omega t}\bigg|_0^1 - \int_0^1 \left(-\frac{1}{j\omega}e^{-j\omega t}\right)\mathrm{d}t \\
&= -\frac{e^{-j\omega}}{j\omega} - \left[\frac{1}{j^2\omega^2}e^{-j\omega t}\right]\bigg|_0^1 \\
&= -\frac{e^{-j\omega}}{j\omega} + \frac{e^{-j\omega} - 1}{\omega^2}.
\end{aligned}$$