1 Trigonometric Identities

It will be useful to memorize \( \sin \theta, \cos \theta, \tan \theta \) values for \( \theta = 0, \pi/3, \pi/4, \pi/2 \) and \( \pi \pm \theta, 2\pi - \theta \) for the above values of \( \theta \).

The following identities involving sine and cosine functions will be useful

\[
\begin{align*}
\sin(\theta \pm \phi) &= \sin \theta \cos \phi \pm \cos \theta \sin \phi \\
\cos(\theta \pm \phi) &= \cos \theta \cos \phi \mp \sin \theta \sin \phi \\
\sin \theta \sin \phi &= \frac{1}{2} [\cos(\theta - \phi) - \cos(\theta + \phi)] \\
\cos \theta \cos \phi &= \frac{1}{2} [\cos(\theta - \phi) + \cos(\theta + \phi)] \\
\sin \theta \cos \phi &= \frac{1}{2} [\sin(\theta - \phi) + \sin(\theta + \phi)]
\end{align*}
\]

The following special case of the above formulas are also very useful to commit to memory

\[
\begin{align*}
\cos^2 \theta &= \frac{1}{2} (1 + \cos 2\theta) \\
\sin^2 \theta &= \frac{1}{2} (1 - \cos 2\theta) \\
\cos(2\theta) &= 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta
\end{align*}
\]

Any two numbers \( a \) and \( b \) can be written as \( a = r \cos \theta \) and \( b = r \sin \theta \), where \( r = \sqrt{a^2 + b^2} \) and \( \tan(\theta) = b/a \). If \( a > 0 \), then one also has \( \theta = \tan^{-1} \left( \frac{b}{a} \right) \), where \( \tan^{-1}(x) \) denotes the principal branch of the inverse tangent which satisfies \( \tan^{-1}(\tan(x)) = x \) for \( -\pi/2 < x < \pi/2 \).

The cosine, sine and exponential functions have infinite series (Maclaurin’s series) expansions given by

\[
\begin{align*}
\cos \theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \ldots \\
\sin \theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \ldots \\
e^\theta &= 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} + \ldots
\end{align*}
\]

where \( \theta \) is in radians.
2 Complex Numbers

We will use the letter $j$ to refer to the imaginary number $\sqrt{-1}$. Even though $j$ is not a real number, we can perform all arithmetic operations such as addition, subtraction, multiplication, division with $j$ using the algebra of real numbers.

2.1 Cartesian Form

A complex number $z$ is any number of the form $z = x + jy$, where $x$ is called the real part of $z$ and $y$ is called the imaginary part of $z$. Note: The imaginary part is not $jy$, rather it is only $y$. It is important to stick to this terminology, otherwise computations can go wrong. Often, it is useful to think of a complex number $z = x + jy$ as a vector in a two-dimensional plane as shown in the figure below, where $x$ is the $X$-coordinate and $y$ is $Y$-coordinate of the vector. Due to this relationship between a complex number and the corresponding vector, we will abuse the terminology and use the terms complex number and vector interchangeably, if the context should resolve any possible ambiguity. For example, a complex number is said to lie in the first quadrant (or, second quadrant etc) if the corresponding vector lies in the first quadrant (or, second quadrant etc).

![Complex Number Diagram](image)

2.2 Polar or Exponential Form, Magnitude and Phase

When a complex number is thought of as a vector in two dimensions, the $X$ coordinate $x$ and the $Y$ coordinate $y$ can be expressed in terms of the length of the vector $r$ and the angle made by this vector with the positive $X$-axis, namely $\theta$. Since $x = r \cos \theta$ and $y = r \sin \theta$, $z$ can be expressed as

$$z = r \cos \theta + jr \sin \theta,$$  \hspace{1cm} (5)

where $\theta$ can be in degrees or radians (usually radians) and recall that $2\pi \text{ rad} = 360^\circ$. $r$ is called the magnitude of $z$, denoted by $|z|$ and $\theta$ is called the phase of the complex number $z$, denoted by $\arg z$ or $\angle z$.

Using Euler’s identities $z$ can be written as

$$z = r \cos \theta + jr \sin \theta = re^{j\theta}$$  \hspace{1cm} (6)
This is known as the polar form or exponential form and it is very important to be able to convert a complex number from cartesian form to exponential form and vice versa. It is easy to see that $x, y, r$ and $\theta$ are related according to

$$
x = r \cos \theta, \quad y = r \sin \theta
$$
$$
r = \sqrt{x^2 + y^2}, \quad \tan(\theta) = \frac{y}{x}.
$$

The value of $\theta$ (for all $x, y$) is given by

$$
\theta = \begin{cases}
\arctan\left(\frac{y}{x}\right) & x > 0 \\
\pi + \arctan\left(\frac{y}{x}\right) & y \geq 0, x < 0 \\
-\pi + \arctan\left(\frac{y}{x}\right) & y < 0, x < 0 \\
\frac{\pi}{2} & y > 0, x = 0 \\
-\frac{\pi}{2} & y < 0, x = 0 \\
\text{undefined} & y = 0, x = 0.
\end{cases}
$$

**Example 1.** It is very useful to know the polar form for often used complex numbers such as $1, j, -j, -1$. They are given by $1 = e^{j0}, -1 = e^{j\pi}, j = e^{j\pi/2}, -j = e^{-j\pi/2}$.

Caution: $r$ must be positive in the above expression. For example, if $z = -2e^{j\pi/4}$, we must rewrite $z$ as $z = 2e^{j\pi/4}$ and interpret $r$ as $2$ instead of $-2$.

Caution: The expression for $\theta$ in (7) does not identify $\theta$ uniquely, since $\tan(\theta) = \frac{y}{x}$ also implies that $\tan(\theta \pm \pi) = \frac{y}{x}$. However, one can use the signs of $(x, y)$ to determine which quadrant contains this vector and then adjust the value of $\theta = \tan^{-1}(y/x)$ accordingly. This is done automatically by the by the more complicated expression for $\theta$.

**Example 2.** Suppose $z_1 = \sqrt{3} + j\frac{1}{2}$ and $z_2 = -\sqrt{3} - j\frac{1}{2}$. It is easy to see that $\tan^{-1}\left(\frac{y_1}{x_1}\right) = \tan^{-1}\left(\frac{y_2}{x_2}\right)$. However, $z_1$ is complex number in the first quadrant, whereas $z_2$ is a complex number is the 3rd quadrant. Therefore, $\theta_1$ should be $\pi/6$ and $\theta_2$ should be $7\pi/6$.

One important aspect of the polar form for a complex number is that adding $2\pi$ to the angle does not change the complex number. Particularly, $re^{j\theta} = re^{j(\theta + 2k\pi)}$. While this seems innocuous at first, this fact will be repeatedly used in the course. An immediate example of where this is useful is given in Section 2.6.

**Example 3.** Express $e^{j2\pi}, e^{-j\pi}, e^{j3\pi}, e^{j9\pi}$ in Cartesian form.

### 2.3 Euler’s Identities

In (4) if one replaces $\theta$ by $j\theta$ and $-j\theta$, we get the following two equations, respectively.

$$
e^{j\theta} = 1 + \frac{j\theta}{1!} + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \ldots
$$
$$
e^{-j\theta} = 1 - \frac{j\theta}{1!} + \frac{(j\theta)^2}{2!} - \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} - \frac{(j\theta)^5}{5!} + \ldots
$$
From (8), (9), (2) and (3) the following relationship can be seen to be true

\[ e^{j\theta} = \cos \theta + j \sin \theta \]
\[ e^{-j\theta} = \cos \theta - j \sin \theta \]
\[ \cos \theta = \frac{1}{2} (e^{j\theta} + e^{-j\theta}) \]
\[ \sin \theta = \frac{1}{2j} (e^{j\theta} - e^{-j\theta}) \]

2.4 Conjugate

The conjugate of a complex number \( z = x + jy \) is given by \( z^* = x - jy \). When \( z \) is written in polar form as \( z = re^{j\theta} \), the complex conjugate is given by \( z^* = re^{-j\theta} \). In general, to compute the conjugate of a complex number, replace \( j \) by \(-j\) everywhere.

![Figure 1: Complex conjugate](image-url)
2.5 Operations on two complex numbers

Let \( z = x + jy = re^{j\theta} \), \( z_1 = x_1 + jy_1 = r_1e^{j\theta_1} \) and \( z_2 = x_2 + jy_2 = r_2e^{j\theta_2} \)

\[
\begin{align*}
  z_1 \pm z_2 & = (x_1 \pm x_2) + j(y_1 \pm y_2) \\
  z_1z_2 & = (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1) = r_1r_2e^{j(\theta_1 + \theta_2)} \\
  zz^* & = x^2 + y^2 = r^2 \\
  |z| & = \sqrt{zz^*} = r \\
  \frac{z_1}{z_2} & = \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2} = \frac{r_1}{r_2}e^{j(\theta_1 - \theta_2)}
\end{align*}
\]

Based on the above operations, the following facts about complex number can be verified.

\[
\begin{align*}
  (z_1 + z_2)^* & = z_1^* + z_2^* \\
  (z_1 z_2)^* & = z_1^* z_2^* \\
  (\frac{z_1}{z_2})^* & = \frac{z_1^*}{z_2^*} \\
  |z_1 - z_2| & = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
  |z_1z_2| & = |z_1||z_2| = r_1r_2 \\
  \frac{|z_1|}{|z_2|} & = \frac{r_1}{r_2}
\end{align*}
\]

2.6 \( n^{th} \) power and \( n^{th} \) roots of a complex number

Let \( z_0 = x_0 + jy_0 = r_0e^{j\theta_0} \). For any integer \( n \), the \( n^{th} \) power of \( z \), \( z^n \) is simply obtained by using (11) \( n \) times. In the polar form, \( z_0^n = r_0^n e^{jn\theta_0} \). Just like how the two real numbers 1 and \( -1 \) have the same square, different complex numbers can have the same \( n^{th} \) power.

Consider the set of distinct complex numbers \( z_k = e^{j\theta_0 + \frac{2\pi k}{n}} \). All the \( z_k \)s are different have the same \( n^{th} \) power for \( k = 0, 1, 2, \ldots, n - 1 \). We can see this by raising \( z_k \) to the \( n^{th} \) power to get

\[
z^n_k = \left( e^{j\theta_0 + \frac{2\pi k}{n}} \right)^n = e^{jn\theta_0 + 2\pi k} = e^{jn\theta_0}.
\]

The \( n^{th} \) root of \( z \) is a bit more interesting and tricky. Any complex number \( z \) which is the solution to the \( n^{th} \) degree equation

\[z^n - z_0 = 0\]

is an \( n^{th} \) root of \( z_0 \). The fundamental theorem of algebra states that an \( n^{th} \) degree equation has exactly \( n \) (possibly complex) roots. Hence, every complex number \( z_0 \) has exactly \( n \), \( n^{th} \) roots. These roots can be found as follows (notice that the use of the fact that \( e^{j\theta} = e^{j(\theta + 2\pi k)} \) is key.

\[
z^n = z_0 \quad \Rightarrow \quad r^n e^{jn\theta} = r_0e^{j\theta_0} = r_0e^{j(\theta_0 + 2k\pi)}
\]

\[
\Rightarrow r = \sqrt[n]{r_0}, \quad \theta = \frac{\theta_0 + 2k\pi}{n} \quad \text{for} \quad k = 0, 1, 2, \ldots n - 1
\]
Clearly, computing \( n \)th roots is much easier in the polar form than in the cartesian form.

**Example 4.** Find the third roots of unity \( \sqrt[3]{1} \)

Since \( 1 = 1e^{j0} \), this corresponds to \( r_0 = 1, \theta_0 = 0 \). Hence, the three roots of unity are given by

\[
r = 1, \quad \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}.
\]

In cartesian coordinates, they are \((1+j0), \left(\frac{-1}{2} + j\frac{\sqrt{3}}{2}\right), \left(\frac{-1}{2} - j\frac{\sqrt{3}}{2}\right)\). These are referred to as \(1, \omega, \omega^2\) sometimes. The three roots are shown in Figure 2.

![Figure 2: Cube roots of unity](image)

### 2.7 Functions of a complex variable

Let \( f(z) \) be a complex function of a complex variable \( z \), i.e., for every \( z \), \( f(z) \) is a complex number. Note that a real number is also considered as a complex number and, hence, \( f(z) \) could have a zero imaginary part. Examples of functions include \( f(z) = |z|, f(z) = \arg(z), f(z) = z^n, f(z) = \exp(z), f(z) = \log(z) \), etc. Both these exponential and logarithmic functions can be interpreted using Euler’s identity as follows.

\[
f(z) = \exp(z) = e^{x}e^{jy} = e^{x} \cos y + je^{x} \sin y
\]
Figure 3: $\Re\{\exp z\}$ vs $\Re\{z\}$

The real part of $f(z)$ is plotted as a function of the real part of $z$, namely $x$ for the case $x < 0$ in Fig. 3.

The logarithm function can be also interpreted using Euler’s identity as $\log(z) = \log(re^{j\theta}) = \log r + j\theta$.

2.8 Complex functions of a real variable

You may be used to dealing with functions of a variable such as $y = f(x)$, where $x$ is called the independent variable and $y$ is called the dependent variable and typically, $y$ takes real values when $x$ takes real values. In this course, we will be interested in complex functions of a real variable such as time or frequency. Such a function, normally denoted as $x(t)$ or $X(\omega)$ is a function which takes a complex value for every real value of the independent variable $t$ or $\omega$. Pay attention to the notation carefully - $t$ or $\omega$ now becomes the independent variable and $x(t)$ or $X(\omega)$ now becomes the dependent variable. We can also think of the complex function as the combination of two real functions of the independent variable, one for the real part of $x(t)$ and one for the imaginary part of $x(t)$.

When dealing with real functions of a real variable, you may be used to plotting the function $x(t)$ as a function of $t$. However, when $x(t)$ is a complex function, there is a problem in plotting this function since for every value of $t$, we need to plot a complex number. In this case, we do one of two things - either we plot the real part of $x(t)$ versus $t$ and plot the imaginary part of $x(t)$ versus $t$, or we plot $|x(t)|$ versus $t$ and $\arg(x(t))$ versus $t$. Either of these is fine, but we do need two plots to effectively understand how $x(t)$ changes with $t$.

**Example 5.** Consider the function $x(t) = e^{j2\pi t} = \cos 2\pi t + j\sin 2\pi t$ for all real values of $t$. This is clearly a complex function of a real variable $t$. $\Re\{x(t)\}$, $\Im\{x(t)\}$, $|x(t)|$, $\arg(x(t))$ are all real functions of the real variable $t$. Hence, we can plot $\Re\{x(t)\}$ versus $t$ or we can plot $\Im\{x(t)\}$ versus $t$ or we can plot $|x(t)|$ versus $t$ and $\arg(x(t))$ versus $t$ as shown in Fig. 4.

**Example 6.** Consider the function $H(\omega) = \frac{1}{1+j\omega}$, where $\omega$ is a real variable. Roughly sketch
Figure 4: Plot of $\Re\{x(t)\}, \Im\{x(t)\}, |x(t)|, \angle x(t)$ versus $t$ for $x(t) = e^{j2\pi t}$.
the magnitude and phase of $H(\omega)$ as a function of $\omega$.

$$H(\omega) = \frac{1}{1 + j\omega}$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + \omega^2}}$$

$$\angle(H(\omega)) = 0 - \tan^{-1}\omega$$

A plot of $|H(\omega)|$ versus $\omega$ and $\angle(H(\omega))$ versus $\omega$ is shown in Fig. 5.

\[\text{Figure 5: Plot of } H(\omega) \text{ vs } \omega \text{ and } \angle(H(\omega)) \text{ versus } \omega \text{ for } H(\omega) = \frac{1}{1 + j\omega}.\]

**Example 7.** Consider the function $X(\omega) = \frac{j\omega}{1 + j\omega}$, where $\omega$ is a real variable. Roughly sketch the magnitude and phase of $X(\omega)$ as a function of $\omega$.

$$X(\omega) = \frac{j\omega}{1 + j\omega}$$

$$|X(\omega)| = \frac{\omega}{\sqrt{1 + \omega^2}}$$

$$\angle(X(\omega)) = \begin{cases} -\frac{\pi}{2} - \tan^{-1}\omega, & \omega < 0 \\ \frac{\pi}{2} - \tan^{-1}\omega, & \omega > 0 \end{cases}$$

A plot of $|X(\omega)|$ versus $\omega$ and $\angle(X(\omega))$ versus $\omega$ is shown in Fig. 6.
2.9 Examples

1. Let $z_1 = 2e^{j\pi/4}$ and $z_2 = 8e^{j\pi/3}$. Find
   a) $2z_1 - z_2$
   b) $\frac{1}{z_1}$
   c) $\frac{z_1}{z_2}$
   d) $\sqrt{z_2}$

2. What is $j^j$?

3. Let $z$ be any complex number. Is it true that $(e^z)^* = e^{z^*}$?

4. Plot the magnitude and phase of the function $X(f) = e^{j\pi f} + e^{3j\pi f}$, for $-1 \leq f \leq 1$.

5. Prove that
   $$\int e^{ax} \cos(bx) \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos(bx) + b \sin(bx))$$

2.10 References

3 Geometric Series

A series of the form $ab^k, ab^{k+1}, \ldots, a^{k+l}$, where $a$ and $b$ can be any complex number is called a geometric series with $l + 1$ terms. For example, $1, \frac{1}{2}, \frac{1}{4}, \ldots$ is an infinite geometric series with $a = 1$, $b = \frac{1}{2}$. You may have seen these before, but in this class often we will be interested in the case when $b$ (and $a$) are complex numbers. Luckily, nothing changes from when $a$ and $b$ are just real numbers.

We will particularly be interested in writing a closed form expression for the sum of consecutive terms of a geometric series. The most general result that you should memorize is that

$$\sum_{n=k}^{l} a b^n = \begin{cases} a \left( \frac{b^k - b^{l+1}}{1-b} \right), & b \neq 1; \\ a(l - k + 1), & b = 1. \end{cases} \quad (13)$$

A few special cases of the above general result are important. Just convince yourself that these are true

$$\sum_{n=k}^{\infty} a b^n = a \left( \frac{b^k}{1-b} \right), \quad |b| < 1;$$

$$\sum_{n=0}^{\infty} a b^n = a b^{-k} \left( \frac{b}{b-1} \right), \quad |b| > 1;$$

Another useful result is

$$\sum_{n=0}^{\infty} n b^n = \frac{b}{(1-b)^2}, \quad |b| < 1 \quad (14)$$

Here are couple of examples to try out

1. For any two given integers $k$ and $M$, what is $\sum_{n=0}^{M-1} e^{i \omega k n}$?

2. Just for intellectual curiosity - Can you prove the results in (13) and (14)?

4 Integration by Parts

Let $u(x)$ and $v(x)$ be differentiable and either $u'(x)$ or $v'(x)$ be continuous. Then,

$$\int_{a}^{b} u(x)v'(x)dx = u(x)v(x) \bigg|_{a}^{b} - \int_{a}^{b} u'(x)v(x)dx.$$

**Example 8.** For any $\omega$, consider the integral

$$\int_{0}^{1} te^{-j\omega t}dt$$
and let \( u(t) = t \) and \( v'(t) = e^{-j\omega t} \). Then, \( u'(t) = 1 \) and \( v(t) = -\frac{1}{j\omega}e^{-j\omega t} \). Thus, integration by parts shows that

\[
\int_0^1 te^{-j\omega t} \, dt = -\frac{t}{j\omega}e^{-j\omega t}\bigg|_0^1 - \int_0^1 \left(-\frac{1}{j\omega}e^{-j\omega t}\right) \, dt
\]

\[
= -\frac{e^{-j\omega}}{j\omega} - \left[\frac{1}{j\omega^2}e^{-j\omega t}\right]\bigg|_0^1
\]

\[
= -\frac{e^{-j\omega}}{j\omega} + \frac{e^{-j\omega} - 1}{\omega^2}.
\]