Spatial Signal Processing

Henry D. Pfister

April 19, 2018

1 Introduction

It is often possible to measure a signal of interest at different spatial locations. Once measured, these multiple signals can be used to improve system performance. For example, modern cell phones and wireless routers typically contain multiple antennas (and receive chains) that sample electromagnetic radiation at multiple points in space. These additional signals are used to provide robustness against interference and multipath fading. Many vehicles also contain microphone arrays for hands-free operation of cellular phones. These arrays improve performance by minimizing the effects of road noise. The underlying principles behind spatial signal processing are largely independent of the physical medium and, thus, the ideas in this handout can applied to many different systems.

2 Amplitude Field of a Isotropic Point Source

An isotropic point source generates an amplitude wave that propagates uniformly in all directions. A good analogy is dropping a small pebble in the middle of a very large lake. The result is a circular ripple that propagates at the same speed in all directions. The response observed at other points in the lake is delayed by an amount of time proportional to the distance from the initial disturbance. Moreover, if multiple pebbles are dropped at many different points in the lake, then the overall response will be given by the sum of the individual responses.

Consider a spatial medium where the speed of propagation is c (in meters per second) and suppose a point source located at $\underline{x}_1 \in \mathbb{R}^3$ generates the signal $s_1(t)$. Then, the response at an arbitrary location $\underline{x} \in \mathbb{R}^3$ is assumed to be

$$s_1(\underline{x}, t) = \frac{1}{\|\underline{x} - \underline{x}_1\|} s_1\left(t - \frac{1}{c}\|\underline{x} - \underline{x}_1\|\right), \tag{1}$$

where $\|\underline{x} - \underline{x}_1\|$ is the Euclidean distance between $\underline{x}$ and $\underline{x}_1$. We note that the decay rate of amplitude with distance follows from assuming lossless wave propagation and conservation of energy. If there are multiple point sources $s_1(t), \ldots, s_N(t)$ located at points $\underline{x}_1, \ldots, \underline{x}_N \in \mathbb{R}^3$, then the overall response at $\underline{x}$ is given by

$$u(\underline{x}, t) = \sum_{n=1}^{N} s_n(\underline{x}, t) = \sum_{n=1}^{N} \frac{1}{\|\underline{x} - \underline{x}_n\|} s_n\left(t - \frac{1}{c}\|\underline{x} - \underline{x}_n\|\right).$$

2.1 Far-Field Approximation and Plane Waves

Although the ripples in the above example always form circles, the local curvature of circle flattens into a straight line as its radius becomes large. Thus, when the distance between a source and measurement is much larger than the distance between other local measurements, we say that the source is in the "far-field". To see this mathematically, we consider a single oscillating source $s_1(t) = \exp(j\omega t)$ and assume $\|\underline{x}_0 - \underline{x}_1\| \gg \|\underline{x}\|$. In the next expression, the notation $O(z)$ represents some function that is upper bounded by a constant

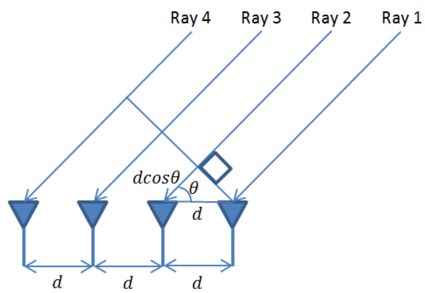

Figure 1: Illustration of uniform linear array

times z as $z \to 0$. Computing the local relative amplitude gives

$$\begin{aligned}
\tilde{s}_1(\underline{x}, t) &= s_1(\underline{x} + \underline{x}_0, t + t_0)/s_1(\underline{x}_0, t_0) \\
&= \frac{\|\underline{x}_0 - \underline{x}_1\|}{\|\underline{x} + \underline{x}_0 - \underline{x}_1\|} \exp\left(j\left(\omega t - \frac{\omega}{c}\|\underline{x} + \underline{x}_0 - \underline{x}_1\|\right)\right) \exp\left(-j\left(\omega t_0 - \frac{\omega}{c}\|\underline{x}_0 - \underline{x}_1\|\right)\right) \\
&= (1 + O(\|\underline{x}\| / \|\underline{x}_0 - \underline{x}_1\|)) \exp\left(j\left(\omega(t - t_0) - \frac{\omega}{c}(\|\underline{x} + \underline{x}_0 - \underline{x}_1\| - \|\underline{x}_0 - \underline{x}_1\|)\right)\right) \\
&= (1 + O(\|\underline{x}\| / \|\underline{x}_0 - \underline{x}_1\|)) \exp\left(j\left(\omega(t - t_0) - \frac{\omega}{c}\left(\frac{(\underline{x}_0 - \underline{x}_1)\cdot \underline{x}}{\|\underline{x}_0 - \underline{x}_1\|} + O(\|\underline{x}\| / \|\underline{x}_0 - \underline{x}_1\|)\right)\right)\right) \\
&\approx \exp\left(j\left(\phi + \omega t - \underline{k}\cdot\underline{x}\right)\right),
\end{aligned}$$

where the unit vector $\underline{n} = (\underline{x}_0 - \underline{x}_1)/\|\underline{x}_0 - \underline{x}_1\|$ is the local *propagation direction* and $\underline{k} = \frac{\omega}{c}\underline{n}$ is the *wave vector* (in units of radians per meter). The quantity $\underline{k}\cdot\underline{x}$ equals the signal phase change (in radians) associated with changing one's local position by $\underline{x}$ meters. Based on this approximation, the response

$$\psi(\underline{x}, t; \omega, \underline{k}) = \exp\left(j\left(\omega t - \underline{k}\cdot\underline{x}\right)\right)$$

is called a *plane wave* with wave vector $\underline{k}$ and angular frequency ω.

2.2 Linear Arrays

Consider a linear array of sensors along the x-axis at points $\underline{y}_m = (dm, 0, 0)$ where $m \in \{-M, -M+1, \ldots, M\}$. With such an array, one can form linear combinations of the received signals to create a spatial filter whose response depends on the spatial location of the source signal. To understand this, we start by computing the response for an arbitrary plane wave $\psi(\underline{x}, t; \omega, \underline{k})$. Due to symmetry, it sufficient to consider the case where the wave vector $\underline{k}$ lies in the xy-plane and meets the x-axis at an angle of θ radians (add figure). In this case, we have $\underline{k}\cdot\underline{y}_m = -\frac{w}{c}\|\underline{n}\|\left\|\underline{y}_m\right\|\cos\theta = -\frac{w}{c}dm\cos\theta$. Denoting the combining weights by $h[-M], h[-M+1], \ldots, h[M]$, the overall response is given by

$$\begin{aligned}
v(t) &= \sum_{m=-M}^{M} h[m] \exp\left(j\left(\omega t - \underline{k}\cdot\underline{y}_m\right)\right) \\
&= \exp(j\omega t) \sum_{m=-M}^{M} h[m] \exp\left(-jm\left(-\frac{w}{c}d\cos\theta\right)\right) \\
&= \exp(j\omega t) H\left(e^{j(-\frac{w}{c}d\cos\theta)}\right),
\end{aligned}$$

where $H(e^{j\kappa})$ is the DTFT of $h[m]$ with respect to the "angular frequency" parameter $\kappa = -\frac{w}{c}d\cos\theta$. The parameter κ can be seen as the phase shift in radians between the signals observed at adjacent sensors.

The spatial response formula also provides some insight into the properties of spatial filtering. For example, the identity $\cos(\theta) = \cos(2\pi - \theta)$ implies that the response will always be the same at the angles θ and $2\pi - \theta$. The formula also shows how one should choose the sensor spacing d. In particular, we observe that $\kappa \in [-\frac{d\omega}{c}, \frac{d\omega}{c}]$. Since $W(e^{j\kappa})$ is periodic with period 2π, the spatial response will repeat (at least partially) if $d\omega/c > \pi$. This is known as spatial aliasing. To avoid spatial aliasing, the critical element spacing corresponds to $d = \frac{\omega}{\pi c} = \frac{\lambda}{2}$, where λ is the signal wavelength of the narrow-band signal.

If $h[m] = u[m+M] - u[m-M-1]$, then we find that

$$
\begin{aligned}
H\left(e^{j\kappa}\right) &= \sum_{m=-M}^{M} e^{-jm\kappa} \\
&= e^{jM\kappa} \sum_{m=0}^{2M} e^{-jm\kappa} \\
&= e^{jM\kappa} \frac{e^{-j(2M+1)\kappa} - 1}{e^{-j\kappa} - 1} \\
&= \frac{\sin\left(\kappa(2M+1)/2\right)}{\sin(\kappa/2)}.
\end{aligned}
$$

The figure below shows and example of the angular response for the case of $M = 5$ and $d\omega/c = 2$.

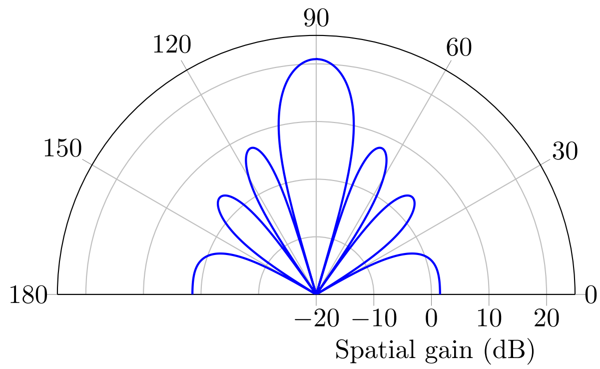

Figure 2: Spatial gain $R(\theta) = \sin((2M+1)\cos\theta)/\sin(\cos\theta)$ versus arrival angle θ (in dB) for a uniformly weighted array with $2M + 1 = 11$ elements and $d\omega/c = 2$.

Since the angular response is given by the DTFT of the weight vector after remapping frequency axis, one can design an arbitrary angular response (for $\theta \in [0, 180°)$) by remapping the desired angular response to the frequency domain and designing FIR filter. The following Matlab code takes this approach and designs a beam pattern centered at $\theta = 45°$ for an array with $2M + 1 = 11$ elements. Notice that a complex FIR filter is designed because complex combining weights are required to prevent a symmetric response about $\theta = 90°$.

```
% Setup parameters
dwc = 2; M = 5;

% Map beam angles (e.g., 30-60 deg) and design complex FIR filter
b = cfirpm(2*M,[-dwc*cos([0 30 35 55 60 180]*pi/180)]/pi,[0 0 1 1 0 0]);

% Compute spatial response by mapping to frequency
th=0:0.01:pi;
h = freqz(b,1,-dwc*cos(th));
```

```
hdb = 20*log10(abs(h));

% Draw polar plot of angular response
polar(th,max(hdb,-20)+20);
yl = get(gca,'YLim');
set(gca, 'YLim', [0 yl(2)]);
```

Example 1. The Matlab example above numerically optimizes the combining weights for a beam centered at $\theta = 45^\circ$ whose width is roughly 30°. Here we compute by hand (e.g., using the window method of filter design) the combining weights for the same situation (i.e., $d\omega/c = 2$ and $2M + 1 = 11$ elements). First, we choose the angular range of $\theta_1 = 35^\circ$ to $\theta_2 = 55^\circ$ and then map these coordinates to the frequencies $\kappa_1 = -\frac{d\omega}{c}\cos\theta_1 = -2\cos(35\pi/180) \approx -1.64$ and $\kappa_2 = -2\cos(55\pi/180) \approx -1.15$. Then, we can design an ideal complex bandpass filter for this frequency range using

$$\begin{aligned} h[m] &= \frac{1}{2\pi}\int_{\kappa_1}^{\kappa_2} e^{j\kappa m} \\ &= \frac{1}{2\pi}\frac{1}{jm}e^{j\kappa m}\bigg|_{\kappa_1}^{\kappa_2} \\ &= \frac{e^{jm(\kappa_1+\kappa_2)/2}}{2\pi jm}\left(e^{j(\kappa_2-\kappa_1)m/2} - e^{-j(\kappa_2-\kappa_1)m/2}\right) \\ &= \frac{e^{jm(\kappa_1+\kappa_2)/2}}{\pi m}\sin\left((\kappa_2-\kappa_1)m/2\right). \end{aligned}$$

Finally, we use a rectangular window to zero all combining weights with $|m| > 5$.

Try plotting the spatial response generated by these weights together with the spatial response of the Matlab-optimized filter from above. Also, try plotting the response of the above weights for a frequency that is 20% higher (i.e., $d\omega/c = 2.4$). Notice that this shifts the angular response as well.

3 Spatial Filtering

3.1 Narrow-Band Signals

The filter designs we have discussed so far all depend on the angular frequency, ω, of the desired signal. If the desired signal has a bandwidth $\Delta\omega$ that is very small in comparison to ω, then the approach described in this section should work well. If not, the next two sections discuss approaches that work for wide-band signals.

Consider the case where there are N different plane wave (i.e., far-field) sources at the same angular frequency ω. Then, the local response is given by

$$u(\underline{x},t) = \sum_{n=1}^{N} a_n \psi(\underline{x},t;\omega,\underline{k}_n) = \sum_{n=1}^{N} a_n \exp\left(j\left(\omega t - \underline{k}_n\cdot\underline{x}\right)\right).$$

We also assume that one can independently measure the response at locations $\underline{y}_1,\ldots,\underline{y}_M$. Then, one can form arbitrary linear combinations of these responses to get

$$
\begin{aligned}
v(t) &= \sum_{m=1}^{M} b_m u(\underline{y}_m, t) \\
&= \sum_{m=1}^{M} b_m \sum_{n=1}^{N} a_n \exp\left(j\left(\omega t - \underline{k}_n \cdot \underline{y}_m\right)\right) \\
&= e^{j\omega t} \sum_{n=1}^{N} a_n \sum_{m=1}^{M} b_m \exp\left(-j\phi_{n,m}\right) \\
&= e^{j\omega t} \underbrace{\begin{bmatrix} a_1 & a_2 & \cdots & a_N \end{bmatrix}}_{\underline{a}^T} \underbrace{\begin{bmatrix} e^{-j\phi_{1,1}} & e^{-j\phi_{1,2}} & \dots & e^{-j\phi_{1,M}} \\ e^{-j\phi_{2,1}} & e^{-j\phi_{2,2}} & \dots & e^{-j\phi_{2,M}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-j\phi_{N,1}} & e^{-j\phi_{N,2}} & \dots & e^{-j\phi_{N,M}} \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}}_{\underline{b}} \\
&= e^{j\omega t} \underline{a}^T \Phi \underline{b}.
\end{aligned}
$$

where $\phi_{n,m} = \underline{k}_n \cdot \underline{y}_m$ and $\Phi_{n,m} = e^{-j\phi_{n,m}}$.

A common situation is that a desired signal $s_1(t)$ is observed along with multiple interfering signals $s_2(t), \dots, s_N(t)$. In that case, the typical goal is to select the weight vector $\underline{b}$ so that $v(t) = a_1 e^{j\omega t}$ or, if that is not possible, so that $\left|v(t) - a_1 e^{j\omega t}\right|^2$ is minimized. Using the above equation, it is clear that this goal can be achieved with equality if and only if there is a vector $\underline{b}$ such that

$$
\Phi \underline{b} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \triangleq \underline{e}_1.
$$

If equality is not possible, then a good estimate of $s_1(t)$ may still be possible using the "least squares" solution, which finds the vector $\hat{\underline{b}}$ that minimizes $\left\|\Phi\hat{\underline{b}} - \underline{e}_1\right\|$. For example, if Φ has full rank and $M < N$, then this solution is given by

$$
\hat{\underline{b}} = (\Phi^H \Phi)^{-1} \Phi^H \underline{e}_1.
$$

If equality is possible, then there may actually be multiple solutions. In that case, one typically chooses the "minimum-norm" solution $\hat{\underline{b}} = \arg\min_{\underline{b} \in \mathbb{R}^M : \Phi \underline{b} = \underline{e}_1} \|\underline{b}\|$ because this minimizes the effect of noise amplification. For example, if Φ has full rank and $M > N$, then the minimum-norm solution is given by

$$
\hat{\underline{b}} = \Phi^H (\Phi \Phi^H)^{-1} \underline{e}_1.
$$

Example 2. Consider the case where $M = N = 2$, $\underline{y}_1 = (0,0,0)$, $\underline{y}_2 = (1,0,0)$, $\underline{k}_1 = (0, -\sqrt{2}, 0)$, and $\underline{k}_2 = (-1,-1,0)$. Using these parameters, one can compute the matrix

$$
\Phi = \begin{bmatrix} e^{-j0} & e^{-j0} \\ e^{-j0} & e^{-j(-1)} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & e^j \end{bmatrix}.
$$

Since this matrix has determinant $e^{-j} - 1 \neq 0$, it is invertible and we can compute

$$
\underline{b} = \begin{bmatrix} 1 & 1 \\ 1 & e^j \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{e^j - 1} \begin{bmatrix} e^j & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{e^j}{e^j - 1} \\ \frac{1}{1 - e^j} \end{bmatrix} \approx \begin{bmatrix} 0.5 - 0.9152j \\ 0.5 + 0.9152j \end{bmatrix}.
$$

3.2 From Narrow-Band to Wide-Band

Using our knowledge of Fourier transforms, it is conceptually straightforward to extend the previous results for narrow-band systems to wide-band systems. The basic idea is to:

1. Compute the Fourier transform of the signals received at each sensor
2. Solve the single-frequency beamforming problem separately for each frequency
3. Compute the inverse Fourier transform of the resulting signal.

In practice, one would probably use a filter bank based on short-time Fourier anaylsis. This approach allows the spatial response and the frequency response of the sensor array to be designed independently (i.e., one can have a different angular response for each frequency).

3.3 General Setup

In this section, we consider a more general model where the response at sensor m of source n is given by $a_{n,m}s_n(t-\tau_{n,m})$ (cf., (1)). Thus, the overall response at sensor m is given by

$$u_m(t) = \sum_{n=1}^{N} a_{n,m} s_n\left(t - \tau_{n,m}\right).$$

This avoids the far-field approximation. In these notes, we will also switch to discrete time (with sample period $T = 1/F_s$) and assume that each received signal $u_m(t)$ is band-limited to $F_s/2$ Hz (e.g., perhaps by an ideal low-pass filter during reception). Of course, this is equivalent to assuming that each source signal $s_n(t)$ is band-limited to $F_s/2$ Hz. The output of the spatial filter is the linear combination of observed signals through filters $h_1[\ell], \ldots, h_M[\ell]$ (supported on $\ell \in \{\tau_1, \tau_1+1, \ldots, \tau_2\}$) defined by

$$\begin{aligned} v[\ell] &= \sum_{m=1}^{M} \sum_{\tau=\tau_1}^{\tau_2} h_m[\tau] u_m\left((t-\tau)T\right) \\ &= \sum_{m=1}^{M} \sum_{\tau=\tau_1}^{\tau_2} h_m[\tau] u_m[\ell-\tau], \end{aligned}$$

where $u_m[\ell] \triangleq u_m(\ell T)$. Since $v[\ell]$ is a linear function of $h_m[\tau]$ for fixed $u_m[\ell]$, we can restrict our attention to $\ell \in \{\ell_1, \ell_1+1, \ldots, \ell_2\}$ and write this as the matrix equation

$$\underline{v} = U\underline{h},$$

where the vectors $\underline{v}$ and $\underline{h}$ are given by

$$\begin{aligned} \underline{v} &= \left[v[\ell_1], v[\ell_1+1], \ldots, v[\ell_2]\right]^T \\ \underline{h} &= \left[h_1[\tau_1], h_1[\tau_1+1], \ldots, h_1[\tau_2], h_2[\tau_1], \ldots, h_M[\tau_2]\right]^T \end{aligned}$$

and the matrix U is given by

$$U = \begin{bmatrix} u_1[\ell_1-\tau_1] & u_1[\ell_1-\tau_1-1] & \cdots & u_1[\ell_1-\tau_2] & u_2[\ell_1-\tau_1] & \cdots & u_M[\ell_1-\tau_2] \\ u_1[\ell_1+1-\tau_1] & u_1[\ell_1+1-\tau_1-1] & \cdots & u_1[\ell_1+1-\tau_2] & u_2[\ell_1+1-\tau_1] & \cdots & u_M[\ell_1+1-\tau_2] \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ u_1[\ell_2-\tau_1] & u_1[\ell_2-\tau_1-1] & \cdots & u_1[\ell_2-\tau_2] & u_2[\ell_2-\tau_1] & \cdots & u_M[\ell_2-\tau_2] \end{bmatrix}.$$

The primary question becomes how should one choose the values of the combining filters $h_m[\tau]$ (or equivalently $\underline{h}$) in order to reconstruct $s_1[\ell]$. In that regard, a reasonable choice is to minimize the power in the error signal

$$e[\ell] = v[\ell] - s_1[\ell].$$

In vector notation, we define

$$\underline{s} = [s_1[\ell_1], s_1[\ell_1+1], \ldots, s_1[\ell_2]]^T$$
$$\underline{e} = [e[\ell_1], e[\ell_1+1], \ldots, e[\ell_2]]^T$$

so that $\underline{e} = U\underline{h} - \underline{s}$. Then, minimizing the power in the error signal is equivalent to solving the linear least-squares minimization problem given by

$$\min_{\underline{h} \in \mathbb{R}^{M(\tau_2-\tau_1+1)}} \left\| U\underline{h} - \underline{s} \right\|^2 .$$

In general, the matrix U should be full rank as long as $\ell_2 - \ell_1 + 1 \gg M(\tau_2 - \tau_1 + 1)$ and there is some noise in the system. If U is full-rank, then the unique solution is given by

$$\hat{\underline{h}} = (U^T U)^{-1} U^T \underline{s}.$$

Solutions computed using this method automatically adjust the spatial-frequency response to optimize the SNR of the desired signal. That means that both the spectral and spatial characteristics of the signal are exploited. The main practical issue is that the solution above requires advance knowledge of the received signals and the desired signal.

4 Applications

Fortunately, the values of the $\underline{h}$ vector actually depend only on the correlation structure between all the signals involved. Thus, one would get the same answer if all source signals were replaced by random noise with the same frequency content. Thus, one can solve for the combining filters using test signals whose power spectral densities are similar to what one expects in practice.

When there are sufficiently many microphones, there are also more advanced techniques that can adaptively learn good combining filters from the observed data.