Derivative Trick for Geometric Sums

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Assuming $|x| < 1$, the geometric series is given by

$$G(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

Differentiate once to get

$$\begin{aligned} G'(x) &= \sum_{k=0}^{\infty} \frac{d}{dx} x^k = \sum_{k=1}^{\infty} k\, x^{k-1} \\ &= \frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}. \end{aligned}$$

Differentiate twice to get

$$\begin{aligned} G''(x) &= \sum_{k=0}^{\infty} \frac{d^2}{dx^2} x^k = \sum_{k=1}^{\infty} k(k-1)\, x^{k-2} \\ &= \frac{d}{dx} G'(x) = \frac{d}{dx} \frac{1}{(1-x)^2} = \frac{2}{(1-x)^3}. \end{aligned}$$

Finally, use $k^2 = k(k-1) + k$ and multiply the second sum by x to get

$$\begin{aligned} \sum_{k=1}^{\infty} k^2 x^{k-1} &= x \sum_{k=1}^{\infty} k(k-1) x^{k-2} + \sum_{k=1}^{\infty} k x^{k-1} \\ &= \frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2} \\ &= \frac{1+x}{(1-x)^3}. \end{aligned}$$

For a geometric random variable X with $\Pr(X = k) = p(1-p)^{k-1}$, $x = 1-p$ gives the variance

$$E[X^2] - E[X]^2 = \left(p \frac{1+x}{(1-x)^3} - \left(p \frac{1}{(1-x)^2} \right)^2 \right) \Bigg|_{x=1-p} = p \frac{2-p}{p^3} - \frac{1}{p^2} = \frac{1-p}{p^2}.$$