In Class Problems

Problems:

1. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 4, 9, 11, 12\}$, $C = \{3, 5, 7, 8, 10, 11\}$, and $D = \{2, 4, 6, 7, 8, 9, 10\}$. Taking the universal set to be $\Omega = A \cup B \cup C \cup D$, find
   (a) $A \cup B$
   (b) $A \cap B$
   (c) $(A \cup B)^c$
   (d) $(A \cup B) \cap C$
   (e) $(A \cap B) \cup D^c$
   (f) $(A \cap C) \cup (D \cap C)$

2. A coin is tossed twice. Alice claims that the event of two heads is at least as likely if we know that the first toss is a head than if we know that at least one of the tosses is a head. Is she right? Does it make a difference if the coin is fair or unfair? How can we generalize Alice's reasoning?

3. Fix two sets A and C. If $C \subset A$, show that for every set B,
$$(A \cap B) \cup C = A \cap (B \cup C). \tag{1}$$
   Also show that if (1) holds for some set B, then $C \subset A$ (and thus (1) holds for all sets B).

4. Show each of the following identities using Venn diagrams.
   (a) $A \cap B \subseteq A \cup B$.
   (b) $A = (A \cap B) \cup (A - B)$.
   (c) $A \cup B = A \cup (B - A)$.
   (d) If $C \subseteq A$, then $(A \cap B) \cup C = A \cap (B \cup C)$.

5. Consider rolling a six-sided die. Let A be the set of outcomes where the roll is an even number. Let B be the set of outcomes where the roll is greater than 3. Calculate and compare the sets on both sides of De Morgan's laws
$$(A \cup B)^c = A^c \cap B^c, \qquad (A \cap B)^c = A^c \cup B^c.$$

6. (a) How many different 7-place license plates are possible if the first 2 places are for letters and the other 5 for numbers?
   (b) Repeat part (a) under the assumption that no letter or number can be repeated in a single license plate.

7. Show that the probability of the union of two events can be generalized to three events as follows:
$$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(A \cap C) \\ - \Pr(B \cap C) + \Pr(A \cap B \cap C).$$

8. **De Méré's puzzle.** A six-sided die is rolled three times independently. Which is more likely: a sum of 11 or a sum of 12? (This question was posed by the French nobleman de Méré to his friend Pascal in the 17th century.)

9. Suppose that there are n students in a room, all born in 1985 (a non-leap year). What is the probability that no two of them celebrate their birthday on the same day of the year? How large does n need to be so that this probability becomes less than 0.5?

10. After playing 5-card stud and 7-card stud for a while, a group of poker players decide to try 6-card stud. In this game, each player is dealt a hand of 6 cards from a standard 52 card deck and the player with the highest 5-card poker hand wins.

    (a) How many distinct hands are there?

    (b) How many hands are ranked "three-of-a-kind"?

    (c) How many hands make a "full house"? (Hint: this includes hands with two triplets)

11. Consider an experiment whose sample space consists of a countably infinite number of points. Show that not all points can be equally likely. Can all points have positive probability of occurring?

12. If it is assumed that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt

    (a) a flush? (A hand is said to be a flush if all 5 cards are of the same suit.)

    (b) one pair? (This occurs when the cards have denominations a, a, b, c, d, where a, b, c, and d are all distinct.)

    (c) two pairs? (This occurs when the cards have denominations a, a, b, b, c, where a, b, and c are all distinct.)

    (d) three of a kind? (This occurs when the cards have denominations a, a, a, b, c, where a, b, and c are all distinct.)

    (e) four of a kind? (This occurs when the cards have denominations a, a, a, a, b.)

13. Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other is either a ten, a jack, a queen, or a king?

14. In a certain community, 36 percent of the families own a dog, and 22 percent of the families that own a dog also own a cat. In addition, 30 percent of the families own a cat. What is

    (a) the probability that a randomly selected family owns both a dog and a cat;

    (b) the conditional probability that a randomly selected family owns a dog given that it owns a cat?