In Class Problem Set 2

Problems:

1. **Using a biased coin to make an unbiased decision.** Alice and Bob want to choose between the opera and the movies by tossing a fair coin. Unfortunately, the only available coin is biased and lands heads with probability p. Since they both know p, whomever calls the toss in the air can bias the decision towards their preference. Design a coin-toss experiment, which does not depend on p, so that they can use the biased coin to make a decision so that either option (opera or movies) is equally likely?

2. From a group of n people, suppose that we want to choose a committee of k, $k \leq n$, one of whom is to be designated as chairperson.

   (a) By focusing first on the choice of the committee and then on the choice of the chair, argue that there are $\binom{n}{k}k$ possible choices.

   (b) By focusing first on the choice of the non chair committee members and then on the choice of the chair, argue that there are $\binom{n}{k-1}(n-k+1)$ possible choices.

   (c) By focusing first on the choice of the chair and then on the choice of the other committee members, argue that there are $n\binom{n-1}{k-1}$ possible choices.

   (d) Conclude from parts (a), (b), and (c) that

   $$k\binom{n}{k} = (n-k+1)\binom{n}{k-1} = n\binom{n-1}{k-1}.$$

   (e) Use the factorial definition of $\binom{m}{k}$ to verify the identity in part (d).

3. Two balls are chosen randomly from an urn containing 8 white, 4 black, and 2 orange balls. Suppose that we win \$2 for each black ball selected and we lose \$1 for each white ball selected. Let X denote our winnings. What are the possible values of X, and what are the probabilities associated with each value?

4. Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times. What are the possible values of X?

5. Suppose the arrival of telephone calls at a switch can be modeled with a Poisson distribution. That is, if X is the number of calls that arrives in t minutes then

   $$\Pr(X = k) = \frac{(\lambda t)^k}{k!}e^{-\lambda t}, \quad k = 0, 1, 2, \ldots$$

   where λ is the average arrival rate in calls/minute. Suppose that the average rate of calls is 10 per minute.

   (a) What is the probability that fewer than three calls will be received in the first 6 seconds?

   (b) What is the probability that fewer than three calls will be received in the first 6 minutes?

6. An internet service provider uses 50 dial-up modems to serve the needs of 1000 customers. It is estimated that at a given time, each customer will need a connection with probability 0.01, independently of the other customers.

   (a) What is the PMF of the number of modems in use at a given time?

   (b) Repeat part (a) by approximating the PMF of the number of customers that need a connection with a Poisson PMF.

   (c) What is the probability that there are more customers needing a connection than there are modems? Provide an exact, as well as an approximate formula based on the Poisson approximation of part (b).