Derivation of the Poisson CDF Formula using the Regularized Incomplete Gamma Function

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1 Introduction

This document provides a proof of the well-known identity that relates the cumulative distribution function (CDF) of a Poisson random variable to the regularized upper incomplete gamma function. For a Poisson random variable X with rate parameter $\lambda > 0$, the CDF is given by

$$P(X \leq k) = \sum_{i=0}^{k} \frac{\lambda^i e^{-\lambda}}{i!},$$

where k is a non-negative integer. We will show that this is equivalent to the regularized upper incomplete gamma function, defined as

$$Q(s,x) = \frac{\Gamma(s,x)}{\Gamma(s)} = \frac{1}{\Gamma(s)} \underbrace{\int_x^\infty t^{s-1} e^{-t} \, dt}_{:=\Gamma(s,x)}.$$

Specifically, we will prove the following theorem.

Theorem 1. *For a non-negative integer k and a positive real number λ, the cumulative distribution function of a Poisson random variable X with parameter λ is given by*

$$P(X \leq k) = Q(k+1, \lambda).$$

2 Proof via Integration by Parts

Proof. We begin with regularized upper incomplete gamma function $Q(k+1, \lambda)$ which is defined by

$$Q(k+1,\lambda) = \frac{1}{\Gamma(k+1)} \int_\lambda^\infty t^k e^{-t} \, dt.$$

Since k is a non-negative integer, we have $\Gamma(k+1) = k!$, so we can write this as

$$Q(k+1,\lambda) = \frac{1}{k!} \int_\lambda^\infty t^k e^{-t} \, dt.$$

We will evaluate the integral using integration by parts. Let $u = t^k$ and $dv = e^{-t} \, dt$. Then $du = kt^{k-1} \, dt$ and $v = -e^{-t}$. The formula for integration by parts is $\int u \, dv = uv - \int v \, du$. Applying this to our integral, we find that

$$\int_\lambda^\infty t^k e^{-t} \, dt = \left[-t^k e^{-t}\right]_\lambda^\infty - \int_\lambda^\infty (-e^{-t})(kt^{k-1}) \, dt.$$

First, let's evaluate the bracketed term given by

$$\lim_{t\to\infty} (-t^k e^{-t}) - (-\lambda^k e^{-\lambda}) = 0 - (-\lambda^k e^{-\lambda}) = \lambda^k e^{-\lambda}.$$

The limit is zero because the exponential function grows faster than any polynomial. Now substitute this back into our equation to get

$$\int_{\lambda}^{\infty} t^k e^{-t}\,dt = \lambda^k e^{-\lambda} + k\int_{\lambda}^{\infty} t^{k-1}e^{-t}\,dt.$$

Divide both sides by $k!$:

$$\frac{1}{k!}\int_{\lambda}^{\infty} t^k e^{-t}\,dt = \frac{\lambda^k e^{-\lambda}}{k!} + \frac{k}{k!}\int_{\lambda}^{\infty} t^{k-1}e^{-t}\,dt.$$

The left side is $Q(k+1,\lambda)$. For the right side, we can simplify the fraction and rewrite the integral in terms of the regularized gamma function,

$$Q(k+1,\lambda) = \frac{\lambda^k e^{-\lambda}}{k!} + \frac{1}{(k-1)!}\int_{\lambda}^{\infty} t^{k-1}e^{-t}\,dt.$$

The integral term is $Q(k,\lambda)$, so we have the recurrence relation given by

$$Q(k+1,\lambda) = \frac{\lambda^k e^{-\lambda}}{k!} + Q(k,\lambda).$$

We can apply this relationship repeatedly, starting from k and going down to 0 to get

$$\begin{aligned}
Q(k+1,\lambda) &= \frac{\lambda^k e^{-\lambda}}{k!} + Q(k,\lambda)\\
&= \frac{\lambda^k e^{-\lambda}}{k!} + \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!} + Q(k-1,\lambda)\\
&= \dots\\
&= \frac{\lambda^k e^{-\lambda}}{k!} + \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!} + \dots + \frac{\lambda^1 e^{-\lambda}}{1!} + Q(1,\lambda).
\end{aligned}$$

To complete the series, we need to evaluate $Q(1,\lambda)$ with

$$Q(1,\lambda) = \frac{1}{0!}\int_{\lambda}^{\infty} t^0 e^{-t}\,dt = \int_{\lambda}^{\infty} e^{-t}\,dt = \left[-e^{-t}\right]_{\lambda}^{\infty} = (0) - (-e^{-\lambda}) = e^{-\lambda}.$$

Substituting this back into our expanded series gives

$$Q(k+1,\lambda) = \frac{\lambda^k e^{-\lambda}}{k!} + \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!} + \dots + \frac{\lambda^1 e^{-\lambda}}{1!} + \frac{\lambda^0 e^{-\lambda}}{0!}.$$

This can be expressed concisely as the sum

$$Q(k+1,\lambda) = \sum_{i=0}^{k} \frac{\lambda^i e^{-\lambda}}{i!}.$$

By definition, this sum is the CDF of a Poisson random variable with parameter λ at integer k and

$$P(X \leq k) = \sum_{i=0}^{k} \frac{\lambda^i e^{-\lambda}}{i!}.$$

Therefore, we have shown that $P(X \leq k) = Q(k+1,\lambda)$. $\square$