ECE 581: Random Processes

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Random Processes

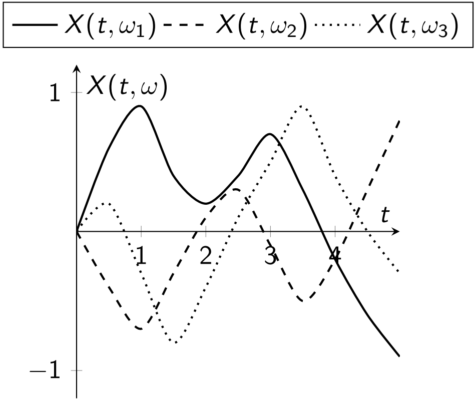
- A random process is a set of RVs on $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\{X(t) : t \in \mathcal{T}\} \quad \text{or} \quad \{X_t : t \in \mathcal{T}\},$$

  where the index t usually denotes time.

- Both notations common for continuous but discrete tends to use subscripts

- For each fixed t, $X(t)$ is a random variable.



Several sample paths $X(t, \omega_i)$.

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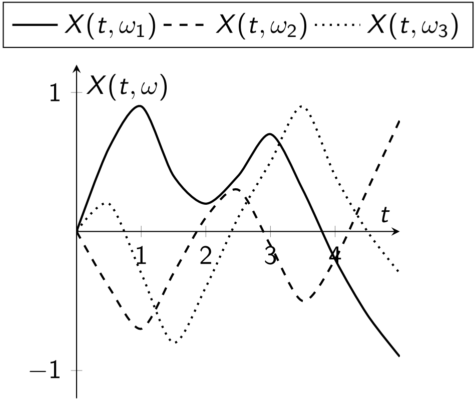
- Both notations common for continuous but discrete tends to use subscripts
- For each fixed t, $X(t)$ is a random variable.
- Experimental outcomes depending on time:
  - measured output of a communication channel,
  - packet arrival times in network,
  - thermal noise in conductors,
  - scores of a sports in a sequence games,
  - daily price of a stock,
  - wealth of a gambler playing over time.



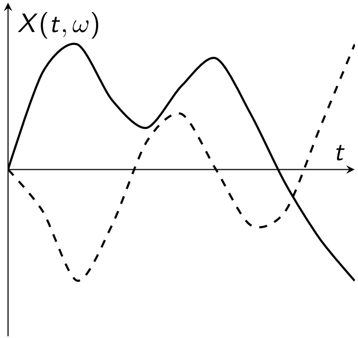
Several sample paths $X(t,\omega_i)$.

Definition of Random Processes

- Formally, a real random process is

$$X : \mathcal{T} \times \Omega \to \mathbb{R}.$$

- For fixed t: $X(t, \cdot)$ is a random variable.
- Fixed outcome ω: $X(\cdot, \omega)$ is a function.



Same process, different outcomes ω.

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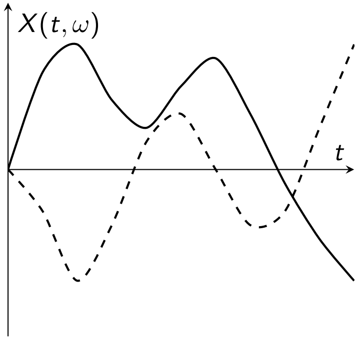
- Fixed outcome ω: $X(\cdot, \omega)$ is a function.
- Sample paths can be continuous, piecewise constant, discrete-time sequences, ...
- One can describe a process in terms of all finite-dimensional distributions of $(X(t_1), \dots, X(t_n))$.

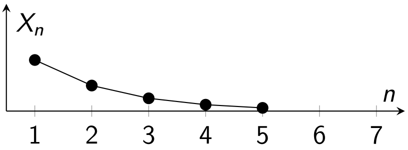


Same process, different outcomes ω.

Discrete-Time Examples

Example 1: $X_n = Z^n$

- $Z \sim \mathrm{Unif}[0,1]$.
- $X_n = Z^n$, $n = 1, 2, \ldots$.
- Each outcome Z gives a decaying sequence.
- First-order cdf: $\mathbb{P}(X_n \le x) = x^{1/n}$.



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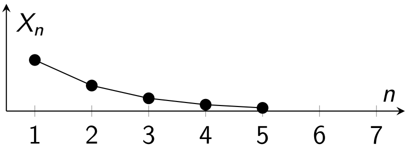
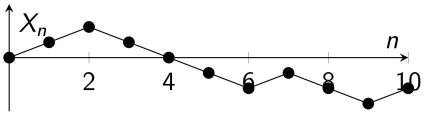


Example 2: Symmetric random walk

- IID steps $Z_n \in \{-1, +1\}$, $\mathbb{P}(Z_n = 1) = \mathbb{P}(Z_n = -1) = \frac{1}{2}$.
- $X_0 = 0$, $X_n = \sum_{i=1}^{n} Z_i$.
- Sample path moves up/down by 1 each step
- $\mathbb{P}(X_n = k) = \binom{n}{(n+k)/2} 2^{-n}$ (parity match).



Discrete-Time Markov Processes

Markov property (discrete time)

- Process $\{X_n\}$ is Markov if

$$\begin{aligned}\mathbb{P}(X_{n+1} = x_{n+1} \mid X_1, \ldots, X_n) \\ = \mathbb{P}(X_{n+1} = x_{n+1} \mid X_n).\end{aligned}$$

- Future depends on past only through X_n.
- IID processes are Markov.
- Random walk $\{X_n\}$ is Markov.

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Independent increments

- $\{X_n\}$ has independent increments if

$$X_{n_1},\ X_{n_2} - X_{n_1}, \ldots, X_{n_k} - X_{n_{k-1}}$$

are independent for all $n_1 < \cdots < n_k$.
- Such processes are Markov.
- Random walk: $X_n - X_{n-1} = Z_n$ are IID $\Rightarrow$ independent increments.
- Converse: Markov $\not\Rightarrow$ independent increments (e.g. AR(1)).

Continuous-Time Examples

Independent increments

- $\{X(t)\}_{t\geq 0}$ has independent increments if for $t_1 < \cdots < t_k$,

  $$X(t_1),\ X(t_2) - X(t_1), \ldots, X(t_k) - X(t_{k-1})$$

  are independent.
- Generalizes the discrete-time notion.
- Key class: counting processes and Brownian motion.

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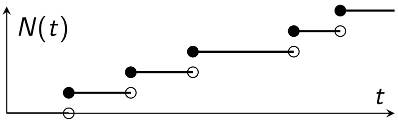
Poisson process (rate λ)

- $N(0) = 0,\ N(t) \in \{0, 1, 2, \ldots\}$.
- Independent increments.
- $N(t) - N(s) \sim \mathrm{Poisson}(\lambda(t - s))$.
- Interarrival times are IID $\mathrm{Exp}(\lambda)$.



Continuous-Time Markov Processes

Markov in continuous time

- $\{X(t)\}$ is Markov if for $s < t$

$$\mathbb{P}(X(t) \in A \mid \{X(u), u \leq s\}) = \mathbb{P}(X(t) \in A \mid X(s)).$$

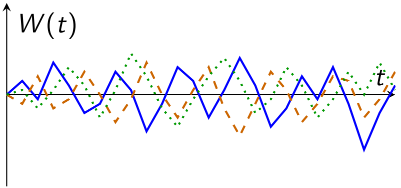
- Future depends on past only through current state.
- Defined via transition probabilities $P_{s,t}(x, A) = \mathbb{P}(X(t) \in A \mid X(s))$.

Independent increments ⇒ Markov

- If increments are independent, conditioning on $X(s)$ shields off the past.

Brownian motion (Wiener process)

- $W(0) = 0$.
- Independent increments.
- $W(t) - W(s) \sim \mathcal{N}(0, t - s)$, $t > s$.
- Sample paths continuous a.s.
- Classic continuous-time Markov process.



Mean and Autocorrelation functions

Definitions: For a process $\{X(t)\}$:

- Mean function

$$\mu_X(t) = \mathbb{E}[X(t)].$$

- Autocorrelation

$$R_X(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)].$$

- Autocovariance

$$k_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2).$$

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Example: random walk

- Z_i IID with $\mathbb{E}[Z_i] = 0$ and $\mathbb{E}[Z_i^2] = 1$
- $X_n = \sum_{i=1}^{n} Z_i$
- Mean:

$$\mu_X(n) = 0.$$

- For $n_1 \leq n_2$,

$$R_X(n_1, n_2) = \mathbb{E}[X_{n_1} X_{n_2}] = n_1.$$

- In general:

$$R_X(m, n) = \min\{m, n\}.$$

Gaussian processes and AR(1) example

Gaussian process

- $X(t)$ is a Gaussian Process if, for any $t_1 < \cdots < t_k$,

$$(X(t_1), \ldots, X(t_k))$$

  is a multivariate Gaussian vector.
- Fully specified by mean $\mu_X(t)$ and autocorrelation $R_X(t_1, t_2)$.
- Examples: white Gaussian noise, Brownian motion, Gauss–Markov processes.

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AR(1) Gauss–Markov process

- $X_1 = Z_1$, $|\alpha| < 1|$
- $X_n = \alpha X_{n-1} + \sqrt{1-\alpha^2} Z_n$
- Z_n IID $\mathcal{N}(0,1)$.
- Linear transform of Gaussian vector $\Rightarrow$ process is Gaussian.
- Mean: $\mu_X(n) = 0$.
- Autocorrelation:

$$R_X(n_1, n_2) = \alpha^{|n_2 - n_1|}.$$

Stationary Processes

Setup

- Assume index set $\mathcal{T}$ is an additive abelian group (e.g. $\mathbb{Z}$ or $\mathbb{R}$).
- Time shifts: $t \mapsto t + \tau$.
- Stationarity = invariance under shifts.

Strict-sense stationarity (SSS)

- For any $t_1, \ldots, t_n$ and τ,

$$(X(t_1), \ldots, X(t_n)) \stackrel{d}{=} (X(t_1+\tau), \ldots, X(t_n+\tau)).$$

- All finite-dimensional distributions invariant under shifts.

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Wide-sense stationarity (WSS)

- Weaker, second-order notion.
- Mean is constant:

$$\mu_X(t) \equiv \mu.$$

- Autocorrelation depends only on time diff:

$$R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau).$$

- For Gaussian processes: SSS $\iff$ WSS.
- Many models in communications assume WSS + Gaussian.