Sequential Experiments and Counting

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ECE 581: Random Signals and Noise

Lecture 2

Slides courtesy of Christ Richmond with slight modifications.

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Remarks

- Last chapter we discussed sequential events and the value of using a probability tree diagram to represent the sample space. Recall that:
  - Tree diagrams capture sequential nature.
  - Probabilities for first branch are prior probabilities.
  - Probabilities of all other subsequent branches are conditional probabilities.
  - Probability of an outcome is obtained by product of all tree probabilities leading to it.
  - The probabilities on the branches leaving any node must sum to 1.
  - The probabilities of all the terminal leaves of the tree must sum to 1.

Examples 2.3 and 2.4 Yates/Goodman

- To be discussed in class

Example 2.3 Problem

Suppose you have two coins, one biased, one fair, but you don't know which coin is which. Coin 1 is biased. It comes up heads with probability 3/4, while coin 2 comes up heads with probability 1/2. Suppose you pick a coin at random and flip it. Let C_i denote the event that coin i is picked. Let H and T denote the possible outcomes of the flip. Given that the outcome of the flip is a head, what is $\mathrm{P}[C_1|H]$, the probability that you picked the biased coin? Given that the outcome is a tail, what is the probability $\mathrm{P}[C_1|T]$ that you picked the biased coin?

Example 2.4 Problem

In the Monty Hall game, a new car is hidden behind one of three closed doors while a goat is hidden behind each of the other two doors. Your goal is to select the door that hides the car. You make a preliminary selection and then a final selection. The game proceeds as follows:

- You select a door.
- The host, Monty Hall (who knows where the car is hidden), opens one of the two doors you didn't select to reveal a goat.
- Monty then asks you if you would like to switch your selection to the other unopened door.
- After you make your choice (either staying with your original door, or switching doors), Monty reveals the prize behind your chosen door.

To maximize your probability P[C] of winning the car, *is switching to the other door either (a) a good idea, (b) a bad idea or (c) makes no difference?*

Counting: Permutations and Combinations

- An exhaustive list of mutually exclusive events is called the sample space.
- Counting possibilities is an important exercise for various processes.
- Consider choosing a subset of k items from a total list of n distinct items.

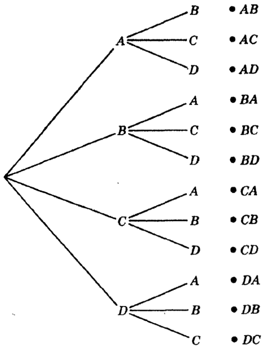
- How many possible k-length sequences are there? The answer is that it depends:
  - If each item is replaced after choosing it from the total list of n possible items, then each selection will have n possibilities. Thus, the total number of k-length sequences is
  $$n \times n \times \cdots \times n \text{ (i.e. } k \text{ times)} = n^k.$$
  - If, however, each item can only be chosen once without replacement, then the first item will have n possibilities; the second will have $n-1$ possibilities; the third will have $n-2$ possibilities, etc.; and the k-th item will be chosen from the remaining $n-(k-1)$ items. Thus, the total number of possible k-length sequences is
  $$n \times (n-1) \times (n-2) \times \cdots \times (n-(k-2)) \times (n-(k-1)) = \frac{n!}{(n-k)!} \triangleq (n)_k$$
  where $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1.$

Counting: Permutations and Combinations

- How many ways are their to select k distinguishable items in order from n?
- This is an interpretation of the $n!$ divided by $(n-k)!$
  - $n!$ is the total number of possible arrangements for an n-length sequence.
  - For any fixed set of values for the first k items in the n-length sequence, note that the last $(n-k)$ items can be in $(n-k)!$ different orders.
  - Thus, by taking the ratio $n!/(n-k)!$ we obtain simply the total number of possible k-length sequences.

Example: Counting of Permutations and Combinations

- How many ways can $k = 2$ of $n = 4$ items A, B, C, D be chosen with replacement?
  - The answer is $n^k = 4^2 = 16$.
  - A tree diagram can be used to keep track as in figure: the first item has four choices indicated by the first four branches emerging from the initial node. The second item only has 3 choices. Thus, the total number of $k = 2$ length sequences is $4 \times 3 = 12$; namely,



$$\Longrightarrow \quad \begin{matrix} AB & BA & CA & DA \\ AC & BC & CB & DB \\ AD & BD & CD & DC \end{matrix}$$

Example: Counting of Permutations and Combinations

Continuing discussion of example: $n = 4$ items A, B, C and D

- Regarding interpretation of the $n!$ divided by $(n-k)!$, note that the total number of $n = 4$ length sequences is $n! = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.
  - Can you see how to modify the tree diagram to show all 24 sequences?
- For any fixed set of values for the first $k = 2$ items in a $n = 4$ length sequence, the last $(n-k) = (4-2) = 2$ items have $(n-k)! = 2! = 2$ possibilities.
  - For example, all 4-length sequences starting with AB are given by $\{ABCD, ABDC\}$;
  - All 4-length sequences starting with CB are given by $\{CBAD, CBDA\}$; etc.
  - By taking the ratio $n!/(n-k)! = 24/2 = 12$ we obtain the precise number of $n =$ 4-length sequences whose beginning $k = 2$ letters yield the set of all $k =$ 2-length sequences once.

Example: Counting of Permutations and Combinations

- Continuing the discussion of $n = 4$ items A, B, C and D:
  - Our sequence list counts all arrangements (or permutations), i.e. we're treating sequence AB as different from BA, and sequence AC as different from CA, and so on; i.e.

$$\#\ k\text{-length permutations of } n \text{ items} = \frac{n!}{(n-k)!}.$$

  - Suppose we don't care about the order, but only the combination (or set) of items, where sequences such as AB and BA are treated as the same outcome?
  - Given k items there are $k!$ such possible arrangements. Thus, a count of the total number of k-length combinations is obtained by dividing the total number of k-length permutations, i.e. $n!/(n-k)!$, by the number of possible ways to arrange k items, i.e. $k!$:

$$\#\ k\text{-length combinations of } n \text{ items} = \frac{n!}{k! \cdot (n-k)!} \triangleq \binom{n}{k}$$

For $n = 4$ and$k = 2$, $\binom{4}{2} = 6$. A set determines its complement and this implies $\binom{n}{k} = \binom{n}{n-k}$.

Example 2.5 Problem

Choose 7 cards at random from a deck of 52 different cards. Display the cards in the order in which you choose them. How many different sequences of cards are possible?

Example 2.10 Problem

There are four queens in a deck of 52 cards. You are given seven cards at random from the deck. What is the probability that you have no queens?

Example 2.11 Problem

There are four queens in a deck of 52 cards. You are given seven cards at random from the deck. After receiving each card you return it to the deck and receive another card at random. Observe whether you have not received any queens among the seven cards you were given. What is the probability that you have received no queens?

Binary Sequences

- An n-length binary sequence is a sequence of binary digits $b_i \in \{0, 1\}$, $i = 1, 2, \ldots, n$, sometimes written as a string $b_1 b_2 \cdots b_n$ or as $(b_1, b_2, \cdots, b_n)$.
  - Binary digits are sometimes referred to as "bits" for short.

- Binary sequences are used often to model various types of sequential experiments where each subexperiment outcome is one of only two possibilities. For example,
  - Series of repeated coin flips, each flip yielding a heads or tails outcome.
  - Series of repeated tests, each test resulting in a pass or fail.
  - Series of repeated attempts, each attempt classified as a success or failure.
  - Series of signal transmissions, each either a $+A$ amplitude signal or $-A$ amplitude.
  - etc.

Binary Sequences: Example 2.17

- **Example 2.17:** For five subexperiments each producing a binary output $b_i \in S_{sub} = \{0, 1\}$, $i = 1, 2, \ldots, 5$ what is the number of observation sequences $(b_1, b_2, b_3, b_4, b_5)$ in which 0 appears twice and 1 appears three times?
  - Trying to list all possibilities and counting will work in this case, but is tedious and not scalable. Let's see if we can reach a solution using what we've learned about counting.
  - Consider one such binary sequence, say 10110. Let's label each digit in the sequence using letters A, B, C, D and E as pointers to keep track:

$$\underbrace{1}_{A}\underbrace{0}_{B}\underbrace{1}_{C}\underbrace{1}_{D}\underbrace{0}_{E}$$

  - First we identify the number of ways these 5 bits can be shuffled around, i.e. determine the number of possible permutations.
    - Clearly, there are $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ possible arrangements.
  - Next we determine how many arrangements point to exactly the same sequence.
    - The three ones have pointers A, C and D. These can be rearranged $3! = 3 \cdot 2 \cdot 1 = 6$ ways, yet yielding the same sequence 10110; namely, ACD, ADC, DAC, DCA, CDA and CAD.

Binary Sequences: Example 2.17 Cont.

Example 2.17 (Cont.):

- More of determining # arrangements pointing to exactly the same sequence:
  - The two zeros have pointers B and E that can be rearranged in $2! = 2 \cdot 1 = 2$ ways, yet yielding the same sequence 10110; namely, BE and EB.
  - Thus, there are $3! \times 2! = 12$ replicas of sequence 10110 among the $5! = 120$ total arrangements.
  - In fact, there are 12 replicas of every sequence containing three 1's and two 0's among the 120 arrangements for exactly the same reason.
  - Dividing by the number of permutations that leave the binary sequence unchanged gives the count of only the distinguishable arrangements or combinations.
  - Let the number subexperiments be denoted $n = 5$, the number of ones be denoted $k = 3$, and then the number of zeros is $n - k = 5 - 3 = 2$.
  - The number of unique sequences with three 1's and two 0's is obtained by

$$\begin{array}{c}\#\text{ unique sequences}\\ \text{with three 1's \& two 0's}\end{array} = \frac{\begin{array}{c}\#\text{ permutations for}\\ n\text{-length sequence}\end{array}}{\begin{array}{c}\#\text{ permutations resulting}\\ \text{in same sequence}\end{array}} = \frac{5!}{3! \times 2!} = \frac{120}{12} = 10,$$

$$\text{in general, } \left(\begin{array}{c}\#\text{ unique sequences}\\ \text{with } k \text{ ones \& } (n-k) \text{ zeros}\end{array}\right) = \frac{n!}{k! \times (n-k)!} = \binom{n}{k}$$

  - The binomial coefficient counts the unique n-length binary sequences with k ones (or $n - k$ zeros).

Binary Sequences: Example 2.17 Cont.

Example 2.17 (Cont.):

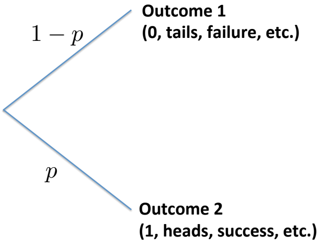
- The 10 unique sequences are

$$\begin{array}{ccccc} 10110 & 11100 & 10011 & 11001 & 00111 \\ 01101 & 01011 & 11010 & 01110 & 10101. \end{array}$$

- Since binary sequences can represent many types of n repeated subexperiments, each having one of two possible outcomes, it is useful to be able to count the number of ways these trials can lead to k ones (or heads, or passes, or successes, etc.).

Bernoulli Process

- Any experiment with two possible outcomes is known as a Bernoulli process. Examples seen already include: coin flip (heads/tails), pass/fail testing, etc.



- Clearly, n repetitions of a Bernoulli process yields an actual or effective binary sequence.
  - Recall that probability is interpreted as a measure of relative frequency (or fraction)
  - Thus, if k is the number of 1's (or heads, successes, etc.) from n independent repetitions of a Bernoulli experiment then as $n \to \infty$ we expect that

$$\frac{k}{n} \to p.$$

Bernoulli Process

- More on n repetitions of a Bernoulli process yielding binary sequence:
  - It is reasonable, therefore, if n is large enough to use relative frequency as an estimate of p, i.e. if p is unknown then it can be estimated via

$$\widehat{p}(b_1, b_2, \ldots, b_n) = \frac{\#\text{ of Outcome 2}}{\#\text{ Repetitions}} = \frac{k}{n}.$$

  - The probability of a specific binary sequence with k ones is $p^k \cdot (1-p)^{n-k}$.
    - (see a probability tree diagram)

Binomial Process

- Consider an $n = 3$-length binary sequence $b_1 b_2 b_3$ resulting from repeated independent Bernoulli subexperiments.
- Let k be the # of ones in this sequence. Clearly, $k \in \mathcal{K} = \{0, 1, 2, 3\}$.
- What is the $\Pr(k = 0)$, $\Pr(k = 1)$, $\Pr(k = 2)$, and $\Pr(k = 3)$?
  - (see probability tree for sample space)
- The sample space is given in the table below.

Outcome	$\Pr(\cdot)$	$k = \#$ones
000	$(1-p)^3$	0
001	$(1-p)^2 p$	1
010	$(1-p)^2 p$	1
011	$(1-p)p^2$	2
100	$(1-p)^2 p$	1
101	$(1-p)p^2$	2
110	$(1-p)p^2$	2
111	p^3	3

$\Longrightarrow$

Clearly, for this example we have

Event $k = (\cdot)$	$\Pr(\cdot)$
0	$(1-p)^3$
1	$3(1-p)^2 p$
2	$3(1-p)p^2$
3	p^3

Binomial Process

- Since $\Pr(k \in \mathcal{K}^c) = 0$, and $k \in \mathcal{K}$ for every sequence $b_1 b_2 b_3$, it follows that

$$\begin{aligned} &\Pr(k=0) + \Pr(k=1) + \Pr(k=2) + \Pr(k=3) \\ &= (1-p)^3 + 3(1-p)^2 p + 3(1-p)p^2 + p^3 = 1 \end{aligned} \tag{1}$$

- Setting $z_1 = p$, $z_2 = 1-p$, and $n = 3$ in binomial formula we also obtain (1), i.e.

$$\begin{aligned} (z_1 + z_2)^n &= \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k} \\ (p + 1 - p)^3 &= \sum_{k=0}^{3} \binom{n}{k} p^k (1-p)^{3-k} \\ 1 &= (1-p)^3 + 3(1-p)^2 p + 3(1-p)p^2 + p^3. \end{aligned}$$

  Hence, the name "Binomial process" for k.

- For the length-n binary sequence $b_1 b_2 \cdots b_n$ resulting from repeated independent Bernoulli subexperiments, the probability of $k \in \mathcal{K} = \{0, 1, \ldots, n\}$ ones in this sequence equals

$$\Pr(k \text{ ones}) = \binom{n}{k} p^k (1-p)^{n-k}$$

Example 2.19 Problem

What is the probability $\mathrm{P}[E_{2,3}]$ of two failures and three successes in five independent trials with success probability p.

Example 1.19 Problem

A company has three machines B_1, B_2, and B_3 making 1 kΩ resistors. Resistors within 50 Ω of the nominal value are considered acceptable. It has been observed that 80% of the resistors produced by B_1 and 90% of the resistors produced by B_2 are acceptable. The percentage for machine B_3 is 60%. Each hour, machine B_1 produces 3000 resistors, B_2 produces 4000 resistors, and B_3 produces 3000 resistors. All of the resistors are mixed together at random in one bin and packed for shipment. What is the probability that the company ships an acceptable resistor?

Example 2.20 Problem

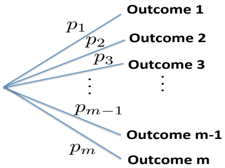
In Example 1.19, we found that a randomly tested resistor was acceptable with probability $P[A] = 0.78$. If we randomly test 100 resistors, what is the probability of T_i, the event that i resistors test acceptable?

Example 2.21 Problem

To communicate one bit of information reliably, cellular phones transmit the same binary symbol five times. Thus the information "zero" is transmitted as 00000 and "one" is 11111. The receiver detects the correct information if three or more binary symbols are received correctly. What is the information error probability P[E], if the binary symbol error probability is $q = 0.1$?

m-ary Sequences

- Consider a trial/subexperiment that can result in only one of m possibilities where $\Pr(\text{Outcome } i) = p_i$, $i = 1, 2, \ldots, m$:



  - This subexperiment will be referred to as m-ary trial and $m = 2$ is a Bernoulli trial.

- Repeating an m-ary trial n times results in a sequence $s_1 s_2 \cdots s_n$ where $s_i \in \{1, 2, \ldots, m\}$.
  - If each subexperiment is independent, then the probability of a specific sequence is

$$\Pr(s_1 s_2 \cdots s_n) = p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m}$$

where $k_j \in \{0, 1, 2, \ldots, n\}$ is the # of subexperiments resulting in outcome $j \in \{1, 2, \ldots, m\}$ such that $k_1 + k_2 + \cdots + k_m = n$.

m-ary Sequences: Example

- Consider nine subexperiments with a ternary output $t_i \in T_{sub} = \{0, 1, 2\}$, $i = 1, 2, \ldots, 9$. How many output sequences $(t_1, t_2, \ldots, t_9)$ contain 3 zeros, 4 ones, and 2 twos?
  - Clearly, $m = 3$, $n = 9$, and interest is in $k_1 = 3$, $k_2 = 4$, and $k_3 = 2$.

- An example sequence with 3 zeros, 4 ones, and 2 twos, is say 210011210:

$$\underbrace{2}_{A}\underbrace{1}_{B}\underbrace{0}_{C}\underbrace{0}_{D}\underbrace{1}_{E}\underbrace{1}_{F}\underbrace{2}_{G}\underbrace{1}_{H}\underbrace{0}_{I}$$

  where letters are used as pointers.

- Note there are 9! total permutations of pointers *ABCDEFGHI*.
  - How many copies of this specific sequence are there among the 9! permutations?
    - The 3 zeros with pointers *CDI* can be arranged $3! = 6$ ways to give the same sequence 210011210.
    - The 4 ones with pointers *BEFH* can be arranged $4! = 24$ ways to give the same sequence 210011210.
    - The 2 twos with pointers *AG* can be arranged $2! = 2$ ways to give the same sequence 210011210.
    - Thus, there are $3! \times 4! \times 2! = 288$ permutations that give exactly the same sequence 210011210.

m-ary Sequences: Example

- Continuing this example on # copies of this specific sequence among 9! permutations:
  - There are 288 copies of every sequence containing 3 zeros, 4 ones, and 2 twos.
  - Dividing $9! = 362880$ by $3! \times 4! \times 2! = 288$ gives the # of unique seq with 3 zeros, 4 ones, and 2 twos:

$$\frac{n!}{k_1! \times k_2! \times k_3!} = \frac{9!}{3! \times 4! \times 2!} = \frac{362880}{288} = 1260.$$

- In general this is called the multinomial coefficient:[1]

$$\frac{n!}{k_1! \times k_2! \times \cdots \times k_m!} \triangleq \binom{n}{k_1, k_2, \ldots, k_m}$$

and is a generalization of the binomial coefficient.

[1] The multinomial theorem provides a compact polynomial expansion of the sum of m terms raised to a power n:

$$(z_1 + z_2 + \cdots + z_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{j=1}^{m} z_j^{k_j}$$

where n is a whole number; the sum is over all m-tuples $(k_1, k_2, \ldots, k_m)$, $k_j \in \{0, 1, 2, \ldots, n\}$ such that $k_1 + k_2 + \cdots + k_m = n$.

Multinomial Process

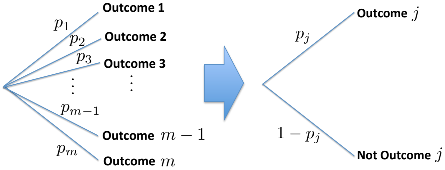
- In general given a n-length sequence $s_1 s_2 s_3 \cdots s_n$ resulting from repeated independent m-ary subexperiments, the probability of obtaining k_j occurences of outcome $j \in \{1, 2, \ldots, m\}$ such that $k_1 + k_2 + \cdots + k_m = n$ is given by

$$\Pr(k_1, k_2, \ldots, k_m) = \underbrace{\binom{n}{k_1, k_2, \ldots, k_m}}_{\substack{\text{Count of all} \\ \text{such sequences}}} \times \underbrace{p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m}}_{\substack{\text{probability of} \\ \text{specific sequence}}}$$

  where clearly the multinomial coefficient is the total number of unique sequences containing k_j occurences of outcome j.
  - This is called the multinomial process due to its connection to the multinomial theorem.
    - Set $z_i = p_i$, $i = 1, 2, \ldots, m$, and note probability space.

A Bernoulli Trial Interpretation of m-ary Trials

- Consider n repeated independent m-ary subexperiments yielding k_j occurences of outcome j such that $k_1 + k_2 + \cdots + k_m = n$. For the possible resulting sequences $s_1 s_2 \cdots s_n$, note that



  i.e. j-th Outcome of m-ary Subexperiment can be interpeted as Bernoulli process.
  - total number of possible arrangements of n-length sequences that have k_1 occurences of outcome 1 is given by the binomial coefficient $\binom{n}{k_1}$; see above Figure with $j = 1$.
  - total number of possible arrangements of the remaining the $n - k_1$ sequence elements having k_2 occurences of outcome 2 is given by the binomial coefficient $\binom{n-k_1}{k_2}$.

A Bernoulli Trial Interpretation of m-ary Trials

- More on repeated m-ary subexperiements:
  - the total number of distinct arrangements of the remaining $n - k_1 - k_2$ sequence elements having k_3 occurences of outcome 3 is given by the binomial coefficient $\binom{n-k_1-k_2}{k_3}$

$$\vdots$$

  - the total number of distinct arrangements of the remaining $n - k_1 - k_2 - \cdots - k_{m-1}$ sequence elements having k_m of outcome m is given by the binomial coefficient $\binom{n-k_1-k_2-\cdots-k_{m-1}}{k_m}$.
  - The product of these binomial decompositions of n subexperiments must yield the same number of permutations as the multinomial coefficient. Thus, we have the identity

$$\binom{n}{k_1, k_2, \ldots, k_m} = \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}\cdots\binom{n-k_1-k_2-\cdots-k_{m-1}}{k_m}$$

  - How might one prove this equality?

References

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[2] D. P. Bertsekas and J. N. Tsitsiklis, *Introduction to Probability*, Athena Sci., 2002.

[3] A. W. Drake, *Fundamentals of Applied Probability Theory*, McGraw-Hill Inc., 1967.