ECE 586: Vector Space Methods
Lecture 11: Frequently Asked Questions

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Here we give a list of questions, and their answers, that were submitted by students after watching the flip video.

1 Linear Transformations

For addition of vector spaces, if my understanding is correct, they need to have the same dimension and number of vectors right? To add 2 subsets U, W of a vector space V, their elements must have the same dimension (since $U+W \triangleq \{\underline{u}+\underline{w} | \underline{u} \in U, \underline{w} \in W\}$, so you need to be able to perform $\underline{u}+\underline{w}$). But, U and W need not have the same number of elements – the definition of the addition of U and W says that you need to perform the sum for all possible pairs $(\underline{u}, \underline{w})$ in $U \times W$.

Also, the slides don't actually consider adding vector spaces. Instead, they define the sum of *subsets of vector spaces*. In this case, the vectors are compatible because they live in the same vector space and vector addition is already defined by that space.

If we use a matrix to represent null space, is it a matrix filled with zeros? There seems to be some confusion about what the nullspace really means. Importantly, for a transformation $T : V \to W$, **the nullspace is a subset of V.** It is precisely the set of things in V that get "killed" by T (i.e. $N(T) \triangleq \{\underline{x} \in V | T(\underline{x}) = 0\}$).

Keeping this in mind, a matrix B representing the null space of a matrix A would probably contain the basis vectors of the null space as rows or columns.

Are vectors in ordered basis orthogonal to each other? Not necessarily, no. Also, we don't yet have a notion of the dot product or orthogonality.

Can you please point out again what is the essential properties to prove the subspace? A subspace X has 2 properties that you should keep in mind:

- It is closed under scalar multiplication, i.e. if $\underline{x} \in X$, then $\forall s \in F, s\underline{x} \in X$ (where F is a field).
- It is closed under vector addition, i.e. if $\underline{x}, \underline{y} \in X$, then $\underline{x} + \underline{y} \in X$.

In slide 5, the rank and nullity are all defined on T and V. What do they do with the space W? Before understanding rank and nullity, it is important to understand range and nullspace. Specifically, for a transformation $T : V \to W$,

- The range $R(T)$ **is a subset of** W, and $\text{rank}(T) = \dim(R(T))$
- The nullspace $N(T)$ **is a subset of** V, and $\text{nullity}(T) = \dim(N(T))$

On slide 1, does the term disjoint mean every vector in the two subspaces are linearly independent? Disjoint just means that the intersection of the sets contains only the $\underline{0}$ vector. You can check that this implies that, if we have 2 disjoint subspaces X, Y, then any $\underline{x} \in X$ cannot be a linear combination of elements in Y (else, $\underline{x}$ would be in Y, which is a contradiction since X and Y are disjoint by assumption).

I am used to the term "orthonormal basis" which means, to my knowledge, essentially the same thing as the definition given for ordered basis. Is there a key difference here I am missing? "Orthonormal" requires you to define an inner product $\langle u, v \rangle$. A basis $\{v_1, ..., v_n\}$ is orthonormal if

$$\langle v_i, v_j \rangle = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$$

Is there a unique coordinate vector for any vector within the space, or are there multiple solutions? An ordered basis $\mathcal{A} = \{v_1, ..., v_n\}$ is linearly independent by definition. Now let's assume there are 2 distinct coordinate vectors for $\underline{x}$ relative to $\mathcal{A}$, $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$. Then, we can write:

$$a_1\underline{v}_1 + ... + a_n\underline{v}_n = b_1\underline{v}_1 + ... + b_n\underline{v}_n \Rightarrow (a_1 - b_1)\underline{v}_1 + ... + (a_n - b_n)\underline{v}_n = \underline{0} \tag{1}$$

By linear independence of $\{v_1, ..., v_n\}$, we get $a_i - b_i = 0 \Rightarrow a_i = b_i, \ \forall i \in \{1, ..., n\}$. Hence the coordinate vector of $\underline{x}$ relative to $\mathcal{A}$ is unique.

What applications are there for linear transformations? Common applications include linear regression, neural networks (a sequence of linear operations interspersed with nonlinearities), and Markov chains (e.g., described by a transition probability matrix P which transforms state probability vector after n steps to the state probability vector after $n + 1$ steps).

Other than orthogonal complement, are there any other typical examples of useful direct sum decompositions? "Orthogonality" requires you to define an inner product $\langle u, v \rangle$, which we don't have yet. Also, disjoint subspaces need not be orthogonal.

Does the vector space have to be a direct sum in order for the unique decomposition of a vector to work? Yes, if you want unique decomposition for all vectors. If two subspaces contain the same vector $\underline{x}$, then that vector can be represented either as $\underline{x} + \underline{0}$ or as $\underline{0} + \underline{x}$. Thus, we cannot uniquely determine which parts are in which subspace.