Chapter 2

Metric Spaces and Topology

From an engineering perspective, the most important way to construct a topology on a set is to define the topology in terms of a metric on the set. This approach underlies our intuitive understanding of open and closed sets on the real line. Generally speaking, a metric captures the notion of a distance between two elements of a set. Topologies that are defined through metrics possess a number of properties that make them suitable for analysis. Identifying these common properties permits the unified treatment of different spaces that are useful in solving engineering problems. To gain better insight into metric spaces, we need to review the notion of a metric and to introduce a definition for topology.

2.1 Metric Spaces

A **metric space** is a set that has a well-defined "distance" between any two elements. Mathematically, a metric space abstracts a few basic properties of Euclidean space. Formally, a metric space (X, d) is a set X and a function d called a metric.

Definition 2.1.1. *A* ***metric*** *on a set X is a function*

$$d \colon X \times X \to \mathbb{R}$$

that satisfies the following properties,

1. $d(x, y) \geq 0 \quad \forall x, y \in X$; *equality holds if and only if* $x = y$
2. $d(x, y) = d(y, x) \quad \forall x, y \in X$

3. $d(x,y)+d(y,z)\geq d(x,z) \quad \forall x,y,z\in X$.

Example 2.1.2. *The set of real numbers equipped with the metric of absolute distance* $d(x,y)=|x-y|$ *defines the standard metric space of real numbers* $\mathbb{R}$.

Example 2.1.3. *Given* $\underline{x}=(x_1,\dots,x_n), \underline{y}=(y_1,\dots,y_n)\in\mathbb{R}^n$*, the* ***Euclidean metric*** d *on* $\mathbb{R}^n$ *is defined by the equation*

$$d\left(\underline{x},\underline{y}\right)=\sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}.$$

As implied by its name, the function d *defined above is a metric.*

Problem 2.1.4. *Let* $\underline{x}=(x_1,\dots,x_n), \underline{y}=(y_1,\dots,y_n)\in\mathbb{R}^n$ *and consider the function* ρ *given by*

$$\rho\left(\underline{x},\underline{y}\right)=\max\left\{|x_1-y_1|,\dots,|x_n-y_n|\right\}.$$

Show that ρ *is a metric.*

Problem 2.1.5. *Let* X *be a metric space with metric* d*. Define* $\bar{d}\colon X\times X\to\mathbb{R}$ *by*

$$\bar{d}(x,y)=\min\left\{d(x,y),1\right\}.$$

Show that $\bar{d}$ *is also a metric.*

Let (X,d) be a metric space. Then, elements of X are called **points** and the number $d(x,y)$ is called the **distance** between x and y. Let $\epsilon>0$ and consider the set $B_d(x,\epsilon)=\{y\in X|d(x,y)<\epsilon\}$. This set is called the d**-open ball** (or open ball) of radius ϵ centered at x.

Problem 2.1.6. *Suppose* $a\in B_d(x,\epsilon)$ *with* $\epsilon>0$*. Show that there exists a* d*-open ball centered at* a *of radius* δ*, say* $B_d(a,\delta)$*, that is contained in* $B_d(x,\epsilon)$*.*

One of the main benefits of having a metric is that it provides some notion of "closeness" between points in a set. This allows one to discuss limits, convergence, open sets, and closed sets.

Definition 2.1.7. *A* ***sequence*** *of elements from a set* X *is an infinite list* $x_1,x_2,\dots$ *where* $x_i\in X$ *for all* $i\in\mathbb{N}$*. Formally, a sequence is equivalent to a function* $f\colon\mathbb{N}\to X$ *where* $x_i=f(i)$ *for all* $i\in\mathbb{N}$*.*

Definition 2.1.8. *Consider a sequence $x_1, x_2, \ldots$ of points in a metric space (X, d). We say that x_n **converges** to $x \in X$ (denoted by $x_n \to x$) if, for any $\epsilon > 0$, there is natural number N such that $d(x, x_n) < \epsilon$ for all $n > N$.*

Problem 2.1.9. *For a sequence x_n, show that $x_n \to a$ and $x_n \to b$ implies $a = b$.*

Definition 2.1.10. *A sequence $x_1, x_2, \ldots$ in (X, d) is a **Cauchy sequence** if, for any $\epsilon > 0$, there is a natural number N (depending on ϵ) such that, for all $m, n > N$,*

$$d\left(x_m, x_n\right) < \epsilon.$$

Theorem 2.1.11. *Every convergent sequence is a Cauchy sequence.*

Proof. Since $x_1, x_2, \ldots$ converges to some x, there is an N, for any $\epsilon > 0$, such that $d(x, x_n) < \epsilon/2$ for all $n > N$. The triangle inequality for $d(x_m, x_n)$ shows that, for all $m, n > N$,

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore, $x_1, x_2, \ldots$ is a Cauchy sequence. □

Example 2.1.12. *Let (X, d) be the metric space of rational numbers defined by $X = \mathbb{Q}$ and $d(x, y) = |x - y|$. The sequence $x_1 = 2$, $x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}$ satisfies $x_n \in \mathbb{Q}$ and, using $x_{n+1} - \sqrt{2} = \frac{1}{2x_n}(x_n - \sqrt{2})^2$, one can show it is Cauchy. But, it does not converge in (X, d) because its limit point is the irrational number $\sqrt{2} \notin \mathbb{Q}$.*

2.1.1 Metric Topology

Definition 2.1.13. *Let W be a subset of a metric space (X, d). The set W is called **open** if, for every $w \in W$, there is an $\epsilon > 0$ such that $B_d(w, \epsilon) \subseteq W$.*

Theorem 2.1.14. *For any metric space (X, d),*

1. *$\emptyset$ and X are open*
2. *any union of open sets is open*
3. *any finite intersection of open sets is open*

Proof. This proof is left as an exercise for the reader. □

One might be curious why only finite intersections are allowed in Theorem 2.1.14. The following example highlights the problem with allowing infinite intersections.

Example 2.1.15. *Let $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right) \subset \mathbb{R}$, for $n \in \mathbb{N}$, be a sequence of open real intervals. The infinite intersection*

$$\bigcap_{n\in\mathbb{N}} I_n = \{x \in \mathbb{R} | \forall n \in \mathbb{N}, x \in I_n\} = \{0\}.$$

But, it is easy to verify that $\{0\}$ is not an open set.

Definition 2.1.16. *A subset W of a metric space (X, d) is **closed** if its complement $W^c = X - W$ is open.*

Corollary 2.1.17. *For any metric space (X, d),*

1. *$\emptyset$ and X are closed*
2. *any intersection of closed sets is closed*
3. *any finite union of closed sets is closed*

Sketch of proof. Using the definition of closed, one can apply De Morgan's Laws to Theorem 2.1.14 verify this result. □

Actually, the sets $\emptyset$ and X are both open and closed. Such sets are called **clopen**. For a non-trivial example, consider the standard metric space of rational numbers and choose $W = \{x \in \mathbb{Q} \,|\, x < \sqrt{2}\}$. This set is open because, for all $x \in W$, we have $B(x, \sqrt{2} - x) \subseteq W$. Since $\sqrt{2} \notin \mathbb{Q}$, it follows that $U = \{x \in \mathbb{Q} \,|\, x \geq \sqrt{2}\} = \{x \in \mathbb{Q} \,|\, x > \sqrt{2}\}$ which is open by the same argument. But $U^c = W$, so W is also closed.

Definition 2.1.18. *For any metric space (X, d) and subset $W \subseteq X$, a point $x \in X$ is a **limit point** of W if there is a sequence $w_1, w_2, \ldots \in W$ of distinct elements that converges to x. Equivalently, x is a limit point of W if, for all $\delta > 0$, the set $\{w \in W | d(x, w) < \delta\}$ contains some point besides x.*

Problem 2.1.19. *Prove that the two definitions of a limit point are equivalent.*

By this definition (which is standard), an **isolated point** $w \in W$ is not a limit point because there is no sequence of distinct elements that converges to it. Instead, for any sequence that converges to an isolated point w, there must be an $N \in \mathbb{N}$ such that $w_n = w$ for all $n > N$.

Theorem 2.1.20. *For any metric space (X, d), a subset $W \subseteq X$ is closed if and only if it contains all of its limit points.*

Proof. Assume W is closed and suppose $x \notin W$ (i.e., $x \in W^c$) is a limit point of W. Since W^c is open, there is a $\delta > 0$ such that $B(x, \delta) \subseteq W^c$ and no sequence in W can approach x. This contradicts the supposition that $x \notin W$ is a limit point and implies all limit points must be in W. On the other hand, if W contains all its limit points, then no $x \notin W$ (i.e., $x \in W^c$) can be a limit point. Negating the second definition of a limit point and applying it to any $x \in W^c$, we see that there is a $\delta > 0$ such that $\{w \in W | d(x, w) < \delta\}$ is empty. Thus, $B_d(x, \delta) \subseteq W^c$ and this implies that W^c is open. Thus, W is closed. $\square$

It follows that, for a closed subset $W \subseteq X$, any sequence $w_n \in W$ that converges must converge to a point $w \in W$.

Definition 2.1.21. *For any metric space (X, d) and subset $W \subseteq X$, a point $w \in W$ is in the **interior** of W if and only if there is a $\delta > 0$ such that, for all $x \in X$ with $d(x, w) < \delta$, it follows that $x \in W$.*

Problem 2.1.22. *Prove that the interior of a set is open.*

Definition 2.1.23. *For any metric space (X, d) and subset $W \subseteq X$, a point $x \in X$ is in the **closure** of W if and only if, for all $\delta > 0$, there is a $w \in W$ such that $d(x, w) < \delta$.*

Problem 2.1.24. *Prove that the closure of a set contains all its limit points and, thus, is open.*

The interior of A is denoted by A° and the closure of A is denoted by $\overline{A}$. The **boundary** ∂A of a set A is defined by $\partial A \triangleq \overline{A} \setminus A^\circ$.

2.1.2 Continuity

Let $f\colon X \to Y$ be a function between the metric spaces (X, d_X) and (Y, d_Y).

Definition 2.1.25. *The function f is **continuous** at x_0 if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that, for all $x \in X$ satisfying $d_X(x_0, x) < \delta$,*

$$d_Y\left(f(x_0), f(x)\right) < \epsilon.$$

In precise mathematical notation, one has

$$(\forall\, \epsilon > 0)(\exists\, \delta > 0)(\forall\, x \in \{x' \in X \,|\, d_X(x_0, x') < \delta\}),\; d_Y\left(f(x_0), f(x)\right) < \epsilon.$$

Theorem 2.1.26. *If f is continuous at x_0, then $f(x_n) \to f(x_0)$ for all sequences $x_1, x_2, \ldots \in X$ such that $x_n \to x_0$. Conversely, if $f(x_n) \to f(x_0)$ for all sequences $x_1, x_2, \ldots \in X$ such that $x_n \to x_0$, then f is continuous at x_0.*

Proof. If f is continuous at x_0, then, for any $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x_0), f(x)) < \epsilon$ if $d_X(x_0, x) < \delta$. If $x_n \to x_0$, then there is an $N \in \mathbb{N}$ such that $d_X(x_n, x_0) < \delta$ for all $n > N$. Thus, $d_Y(f(x_0), f(x_n)) < \epsilon$ for all $n > N$ and $f(x_n) \to f(x_0)$.

For the converse, we show the contrapositive. If f is not continuous at x_0, then there exists an $\epsilon > 0$ such that, for all $\delta > 0$, there is an $x \in X$ with $d_X(x_0, x) < \delta$ and $d_Y(f(x_0), f(x)) \geq \epsilon$. For this ϵ and any positive sequence $\delta_n \to 0$, let x_n be the promised x. Then, $x_n \to x_0$ because $d_X(x_0, x_n) < \delta_n \to 0$ but $d_Y(f(x_0), f(x_n)) \geq \epsilon$. Thus, $f(x_n)$ does not converge to $f(x_0)$ for some sequence where $x_n \to x_0$. $\square$

Definition 2.1.27. *The **limit of f at x_0**, $\lim_{x \to x_0} f(x)$, exists and equals $f(x_0)$ if $f(x_n) \to f(x_0)$ for all sequences $x_n \in X$ such that $x_n \to x_0$. Thus, Theorem 2.1.26 implies that the limit of f exists at x_0 if and only if f is continuous at x_0.*

Definition 2.1.28. *The function f is called **continuous** if, for all $x_0 \in X$, it is continuous at x_0. In precise mathematical notation, one has*

$$\begin{aligned}&(\forall\, x_0 \in X)(\forall\, \epsilon > 0)(\exists\, \delta > 0)\\ &\qquad(\forall\, x \in \{x' \in X \,|\, d_X(x_0, x') < \delta\}),\; d_Y\left(f(x_0), f(x)\right) < \epsilon.\end{aligned}$$

Definition 2.1.29. *The function f is called **uniformly continuous** if it is continuous and, for all $\epsilon > 0$, the $\delta > 0$ can be chosen independently of x_0. In precise mathematical notation, one has*

$$(\forall\, \epsilon > 0)(\exists\, \delta > 0)(\forall\, x_0 \in X)$$
$$(\forall\, x \in \{x' \in X \mid d_X(x_0, x') < \delta\}),\ d_Y\left(f(x_0), f(x)\right) < \epsilon.$$

Definition 2.1.30. *A function $f\colon X \to Y$ is called **Lipschitz continuous** on $A \subseteq X$ if there is a constant $L \in \mathbb{R}$ such that $d_Y(f(x), f(y)) \le L d_X(x, y)$ for all $x, y \in A$.*

Let f_A denote the **restriction** of f to $A \subseteq X$ defined by $f_A\colon A \to Y$ with $f_A(x) = f(x)$ for all $x \in A$. It is easy to verify that, if f is Lipschitz continuous on A, then f_A is uniformly continuous.

Problem 2.1.31. *Let (X, d) be a metric space and define $f : X \to \mathbb{R}$ by $f(x) = d(x, x_0)$ for some fixed $x_0 \in X$. Show that f is Lipschitz continuous with $L = 1$.*

2.1.3 Completeness

Suppose (X, d) is a metric space. From Definition 2.1.8, we know that a sequence $x_1, x_2, \ldots$ of points in X converges to $x \in X$ if, for every $\delta > 0$, there exists an integer N such that $d(x_i, x) < \delta$ for all $i \ge N$.

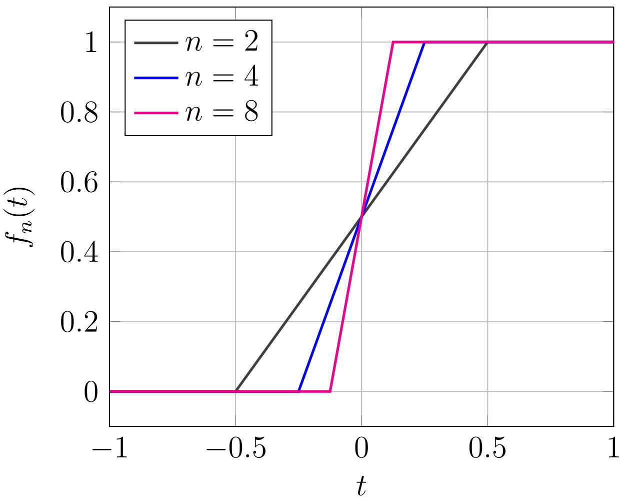

Figure 2.1: The sequence of continuous functions in Example 2.1.32 satisfies the Cauchy criterion. But, it does not converge to a continuous function in $C[-1, 1]$.

It is possible for a sequence in a metric space X to satisfy the Cauchy criterion, but not to converge in X.

Example 2.1.32. *Let $X = C[-1,1]$ be the space of continuous functions that map $[-1,1]$ to $\mathbb{R}$ and satisfy $\|f\|_2 < \infty$, where $\|f\|_2$ denotes the L^2 norm*

$$\|f\|_2 \triangleq \left(\int_{-1}^{1} |f(t)|^2 dt \right)^{\frac{1}{2}}.$$

This set forms a metric space (X, d) when equipped with the distance

$$d(f,g) \triangleq \|f-g\|_2 = \left(\int_{-1}^{1} |f(t)-g(t)|^2 dt \right)^{\frac{1}{2}}.$$

Consider the sequence of functions $f_n(t)$ given by

$$f_n(t) \triangleq \left\{ \begin{array}{ll} 0 & t \in \left[-1, -\frac{1}{n}\right] \\ \frac{nt}{2} + \frac{1}{2} & t \in \left(-\frac{1}{n}, \frac{1}{n}\right) \\ 1 & t \in \left[\frac{1}{n}, 1\right]. \end{array} \right\}$$

Assuming that $m \geq n$, a bit of calculus shows that

$$d(f_n, f_m) = \|f_n(t) - f_m(t)\|_2 = \left(\int_{-1}^{1} |f_n(t) - f_m(t)|^2 dt \right)^{\frac{1}{2}} = \frac{(m-n)^2}{6m^2 n}.$$

Since $m \geq n$, this distance is upper bounded by $\frac{1}{6n}$ and the sequence satisfies the Cauchy criterion. But, it does not converge to a continuous function in $C[-1,1]$.

Definition 2.1.33. *A metric space (X, d) is said to be **complete** if every Cauchy sequence in (X, d) converges to a limit $x \in X$.*

The standard metric space of real numbers with absolute distance is a complete metric space. This fact and other foundational properties of the real numbers can be derived formally using the techniques described below. However, a formal construction of the real numbers will not be provided in these notes.

Example 2.1.34. *Consider the sequence $x_1 = 2$, $x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}$ and observe that $x_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$. We saw earlier that x_n is a Cauchy sequence with limit point $\sqrt{2} \in \mathbb{R}$. But, $\sqrt{2}$ is irrational and thus the rational numbers $\mathbb{Q}$ are not complete.*

Theorem 2.1.35. *A closed subset A of a complete metric space X is itself a complete metric space.*

Proof. Any Cauchy sequence $x_1, x_2, \ldots \in A$ is also a Cauchy sequence in X. This implies that $x_n \to x \in X$ and it follows that $x \in \overline{A}$. Since A is closed, $x \in A$. $\square$

Definition 2.1.36. *An **isometry** is a mapping $\phi\colon X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) that is distance preserving (i.e., it satisfies $d_X(x, x') = d_Y(\phi(x), \phi(x'))$ for all $x, x' \in X$).*

Definition 2.1.37. *A subset A of a metric space (X, d) is **dense** in X if every $x \in X$ is a limit point of the set A. This is equivalent to its closure $\overline{A}$ being equal to X.*

Definition 2.1.38. *The **completion** of a metric space (X, d_X) consists of a complete metric space (Y, d_Y) and an isometry $\phi\colon X \to Y$ such that $\phi(X)$ is a dense subset of Y. Moreover, the completion is unique up to isometry.*

Example 2.1.39. *Consider the metric space $\mathbb{Q}$ of rational numbers equipped with the metric of absolute distance. The completion of this metric space is $\mathbb{R}$ because the isometry is given by the identity mapping and $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.*

Cauchy sequences have many applications in analysis and signal processing. For example, they can be used to construct the real numbers from the rational numbers. In fact, the same approach is used to construct the completion of any metric space.

Definition 2.1.40. *Two Cauchy sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ are equivalent if, for every $\epsilon > 0$, there exists an integer N such that $d(x_k, y_k) \leq \epsilon$ for all $k \geq N$.*

Example 2.1.41. *Let $\mathcal{C}(\mathbb{Q})$ denote the set of all Cauchy sequences $q_1, q_2, \ldots$ of rational numbers where $\sim$ represents the equivalence relation on this set defined above. Then, the set of equivalence classes (or quotient set) $\mathcal{C}(\mathbb{Q})/\sim$ is in one-to-one correspondence with the real numbers. This construction is the standard completion of $\mathbb{Q}$. Since every Cauchy sequence of rationals converges to a real number, the isometry is given by mapping each equivalence class to its limit point in $\mathbb{R}$.*

Definition 2.1.42. *Let A be a subset of a metric space (X, d) and $f\colon X \to X$ be a function. Then, f is a **contraction** on A if $f(A) \subseteq A$ and there exists a constant $\gamma < 1$ such that $d(f(x), f(y)) \leq \gamma d(x, y)$ for all $x, y \in A$.*

Consider the following important results in applied mathematics: Picard's uniqueness theorem for differential equations, the implicit function theorem, and the existence of stationary optimal policies for Markov decision processes. What do they have in common? They each establish the existence and uniqueness of a function and have relatively simple proofs based on the contraction mapping theorem.

Theorem 2.1.43 (Contraction Mapping Theorem)**.** *Let (X, d) be a complete metric space and f be contraction on a closed subset $A \subseteq X$. Then, f has a unique fixed point x^* in A such that $f(x^*) = x^*$ and the sequence $x_{n+1} = f(x_n)$ converges to x^* for any point $x_1 \in A$. Moreover, x_n satisfies the error bounds $d(x^*, x_n) \leq \gamma^{n-1} d(x^*, x_1)$ and $d(x^*, x_{n+1}) \leq d(x_n, x_{n+1})\gamma/(1-\gamma)$.*

Proof. Suppose f has two fixed points $y, z \in A$. Then, $d(y, z) = d\left(f(y), f(z)\right) \leq \gamma d(y, z)$ and $d(y, z) = 0$ because $\gamma \in [0, 1)$. This shows that $y = z$ and any two fixed points in A must be identical.

Since $d(f(x_n), f(x_{n+1})) \leq \gamma d(x_n, x_{n+1})$, induction shows that $d(x_n, x_{n+1}) \leq \gamma^{n-1} d(x_1, x_2)$. Using this, we can bound the distance $d(x_m, x_n)$ (for $m < n$) with

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_n) \\ &\leq \sum_{i=m}^{n-1} d(x_i, x_{i+1}) \leq \sum_{i=m}^{n-1} \gamma^{i-1} d(x_1, x_2) \\ &\leq \sum_{i=m}^{\infty} \gamma^{i-1} d(x_1, x_2) \leq \frac{\gamma^{m-1}}{1-\gamma} d(x_1, x_2). \end{aligned}$$

The sequence x_n is Cauchy because $d(x_m, x_n)$ can be made arbitrarily small (for all $n > m$) by increasing m. As (X, d) is complete, it follows that $x_n \to x^*$ for some $x^* \in X$. Since f is Lipschitz continuous, this implies that $x^* = \lim_n x_n = \lim_n f(x_n) = f(x^*)$ the unique fixed point of f in A.

Arguments similar to the above can be used to prove the stated error bounds. □

Example 2.1.44. *Consider the cosine function restricted to the subset $[0, 1] \subseteq \mathbb{R}$. Since $\cos(x)$ is decreasing for $0 \leq x < \pi$, we have $\cos([0, 1]) = [\cos(1), 1]$ with $\cos(1) \approx 0.54$. The mean value theorem of calculus also tells us that $\cos(y) - \cos(x) = \cos'(t)(y - x)$ for some $t \in [x, y]$. Since $\cos'(t) = -\sin(t)$ and $\sin(t)$ is increasing on $[0, 1]$, we find that $\sin([0, 1]) = [0, \sin(1)]$ with $\sin(1) \approx 0.84$.*

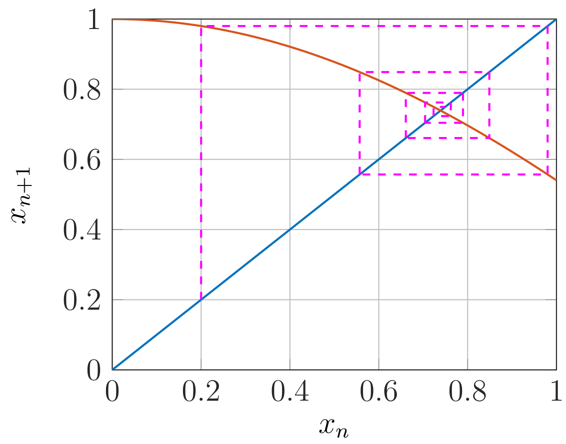

Figure 2.2: Starting from $x_1 = 0.2$, the iteration in Example 2.1.44 maps x_n to $x_{n+1} = \cos(x_n)$. The points are also connected to the slope-1 line to show the path.

Taking the absolute value, shows that $|\cos(y) - \cos(x)| \leq 0.85|y - x|$. *Therefore,* $\cos(t)$ *is a contraction on* $[0, 1]$ *and the sequence* $x_{n+1} = \cos(x_n)$ *(e.g., see Figure 2.2) converges to the unique fixed point* $x^* = \cos(x^*)$ *for all* $x_1 \in [0, 1]$.

2.1.4 Compactness

Definition 2.1.45. *A metric space* (X, d) *is* ***totally bounded*** *if, for any* $\epsilon > 0$, *there exists a finite set of* $B_d(x, \epsilon)$ *balls that cover (i.e., whose union equals)* X.

Definition 2.1.46. *A metric space is* ***compact*** *if it is complete and totally bounded.*

The closed interval $[0, 1] \subset \mathbb{R}$ is compact. In fact, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. On the other hand, the standard metric space of real numbers is not compact because it is not totally bounded.

Theorem 2.1.47. *A closed subset* A *of a compact space* X *is itself a compact space.*

The following theorem highlights one of the main reasons that compact spaces are desirable in practice.

Theorem 2.1.48. *Let* (X, d) *be a compact metric space and* $x_1, x_2, \ldots \in X$ *be a sequence. Then, there is a subsequence* $x_{n_1}, x_{n_2}, \ldots$, *defined by some increasing sequence* $n_1, n_2, \ldots \in \mathbb{N}$, *that converges.*

Proof. We proceed by recursively constructing subsequences $z_n^{(k)}$ starting from $z_n^{(0)} = x_n$. Since X is totally bounded, let $C_k \subset X$ be the centers of a finite set of balls with radius 2^{-k} that cover X (i.e., $\cup_{x \in C_k} B(x, 2^{-k}) = X$). Then, one of these balls (say centered at x') must contain infinitely many elements in $z_n^{(k-1)}$ (i.e., $\exists x' \in C_k,\ |\{n \in \mathbb{N} \mid z_n^{(k-1)} \in B(x', 2^{-k})\}| = \infty$). Next, we extract the subsequence contained in this ball by choosing $z_n^{(k)}$ to be the subsequence of $z_n^{(k-1)}$ contained in $B(x', 2^{-k})$. From the triangle inequality, it follows that $d(y, y') < 2(2^{-k})$ for all $y, y' \in B(x', 2^{-k})$. Thus, $d(z_m^{(k')}, z_n^{(k)}) < 2^{-k+1}$ for all $m > n \geq 1$ and $k' \geq k \geq 1$.

Let $I(k, n)$ be the index in the original sequence associated with $z_n^{(k)}$. Since each stage only removes elements from the previous subsequence and relabels, it follows that $I(k+1, k+1) \geq I(k, k+1) > I(k, k)$. This implies that the sequence $y_k = z_k^{(k)} = x_{I(k,k)}$ is a subsequence of x_n and $d(y_m, y_k) \leq 2^{-k+1}$ for all $m > k$ and $k \geq 1$. Thus, for any $\epsilon > 0$, choosing $N = \lceil \log_2 \frac{1}{\epsilon} \rceil + 1$ shows that y_k is a Cauchy sequence. Since X is complete, it follows that y_k converges to some $y \in X$. $\square$

Functions from compact sets to the real numbers are very important in practice. To keep the discussion self-contained, we first review the extreme values of sets of real numbers. First, we must define the **extended real numbers** $\overline{\mathbb{R}}$ by augmenting the real numbers to include limit points for unbounded sequences $\overline{\mathbb{R}} \triangleq \mathbb{R} \cup \{\infty, -\infty\}$. Using the metric $d_{\overline{\mathbb{R}}}(x, y) \triangleq |\frac{x}{1+|x|} - \frac{y}{1+|y|}|$, this set becomes a compact metric space. The main difference from $\mathbb{R}$ is that, for $x_n \in \overline{\mathbb{R}}$, the statement $x_n \to \infty$ is well defined and equivalent to $\forall M > 0,\ \exists N \in \mathbb{N},\ \forall n > N,\ x_n > M$.

Definition 2.1.49. *The **supremum** (or least upper bound) of $X \subseteq \mathbb{R}$, denoted $\sup X$, is the smallest extended real number $M \in \overline{\mathbb{R}}$ such that $x \leq M$ for all $x \in X$. It is always well-defined and equals $-\infty$ if $X = \emptyset$.*

Definition 2.1.50. *The **maximum** of $X \subseteq \mathbb{R}$, denoted $\max X$, is the largest value achieved by the set. It equals $\sup X$ if $\sup X \in X$ and is undefined otherwise.*

Definition 2.1.51. *The **infimum** (or greatest lower bound) of $X \subseteq \mathbb{R}$, denoted $\inf X$, is the largest extended real number $m \in \overline{\mathbb{R}}$ such that $x \geq m$ for all $x \in X$. It is always well-defined and equals ∞ if $X = \emptyset$.*

Definition 2.1.52. *The **minimum** of $X \subseteq \mathbb{R}$, denoted $\min X$, is the smallest value achieved by the set. It equals $\inf X$ if $\inf X \in X$ and is undefined otherwise.*

Lemma 2.1.53. *Let X be a metric space and $f\colon X \to \mathbb{R}$ be a function from X to the real numbers. Let $M = \sup f(A)$ for some non-empty $A \subseteq X$. Then, there exists a sequence $x_1, x_2, \ldots \in A$ such that* $\lim_n f(x_n) = M$.

Proof. If $M = \infty$, then $f(A)$ has no finite upper bound and, for any $y \in \mathbb{R}$, there exists an $x \in A$ such that $f(x) > y$. In this case, we can let x_1 be any element of A and x_{n+1} be any element of A such that $f(x_{n+1}) > f(x_n) + 1$. In the metric space $(\overline{\mathbb{R}}, d_{\overline{\mathbb{R}}})$, this implies that $d_{\overline{\mathbb{R}}}(x_n, \infty) = |\frac{f(x_n)}{1+|f(x_n)|} - 1| \to 0$ and thus $f(x_n) \to \infty$.

If $M < \infty$, then $f(A)$ has a finite upper bound and, for any $\epsilon > 0$, there is an x such that $M - f(x) < \epsilon$. Otherwise, one arrives at the contradiction $\sup f(A) < M$. Therefore, we can construct the sequence $x_1, x_2, \ldots$ by choosing $x_n \in A$ to be any point that satisfies $M - f(x_n) \leq \frac{1}{n}$. □

Theorem 2.1.54. *Let X be a metric space and $f\colon X \to \mathbb{R}$ be a continuous function from X to the real numbers. If A is a compact subset of X, then there exists $x \in A$ such that $f(x) = \sup f(A)$ (i.e., f achieves a maximum on A).*

Proof. Using Lemma 2.1.53, one finds that there is a sequence $x_1, x_2, \ldots \in A$ such that $\lim_n f(x_n) = \sup f(A)$. Since A is compact, there must also be a subsequence $x_{n_1}, x_{n_2}, \ldots$ that converges. As A is closed, this subsequence must converge to some $x^* \in A$. Finally, the continuity of f shows that

$$\sup f(A) = \lim_n f(x_n) = \lim_k f(x_{n_k}) = f(\lim_k x_{n_k}) = f(x^*).$$ □

Corollary 2.1.55. *Let (X, d) be a metric space. Then, a continuous function from a compact subset $A \subseteq X$ to the real numbers achieves a minimum on A.*

Theorem 2.1.56. *Any bounded non-decreasing sequence of real numbers converges to its supremum.*

Proof. Let $x_1, x_2, \ldots \in \mathbb{R}$ be a sequence satisfying $x_{n+1} \geq x_n$ and $x_n \leq M < \infty$ for all $n \in \mathbb{N}$. Without loss of generality, we can choose the upper bound M to be the supremum $\sup\{x_1, x_2, \ldots\}$. Now, we will prove directly that $x_n \to M$.

First, we note that the definition of the supremum implies that $x_n \leq M$ for all $n \in \mathbb{N}$ and, for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $x_N > M - \epsilon$. Second, since x_n is non-decreasing, this implies that $x_n > M - \epsilon$ for all $n > N$. Third, since $x_n \leq M$ by definition, it follows that $|M - x_n| = M - x_n < \epsilon$ for all $n > N$. Thus, the constructed N satisfies all elements in the definition of $x_n \to M$. □

Lemma 2.1.57. *Let $y_n \in \mathbb{R}$ be a real sequence and $x_n = \sum_{i=1}^{n} y_i$ be its sequence of partial sums. Then, $\sum_{i=1}^{\infty} y_i \triangleq \lim_{n\to\infty} x_n$ exists if and only if the tail of the sum is negligible:*

$$\forall \epsilon > 0,\, \exists N \in \mathbb{N},\, \forall m, n > N,\, \left| \sum_{i=n+1}^{m} y_i \right| < \epsilon.$$

Proof. Since $\mathbb{R}$ is complete, $\lim_{n\to\infty} x_n$ exists if and only if x_n is a Cauchy sequence. Thus, x_n converges if and only if "$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n > N, |x_m - x_n| < \epsilon$". Thus, the result follows from the fact that $|x_m - x_n| = \left|\sum_{i=n+1}^{m} y_i\right|$. □

Lemma 2.1.58. *Let $y_n \in \mathbb{R}$ be a real sequence. If $\sum_{i=1}^{\infty} |y_i| = M < \infty$, then $x_n = \sum_{i=1}^{n} y_i$ satisfies $x_n \to x$ with $|x| < M$.*

Proof. Let $w_n = \sum_{i=1}^{n} |y_i|$ and observe that the following inequality holds,

$$|x_m - x_n| = \left| \sum_{i=n+1}^{m} y_i \right| \le \sum_{i=n+1}^{m} |y_i| = |w_m - w_n|.$$

Since w_n converges, it is Cauchy and the inequality implies that x_m is Cauchy. Thus, x_n converges to some $x \in \mathbb{R}$ and $|x| < M$ follows from $|x_n| \le w_n \le M$. □

2.1.5 Sequences of Functions

Let (X, d_X) and (Y, d_Y) be metric spaces and $f_n \colon X \to Y$ for $n \in \mathbb{N}$ be a sequence of functions mapping X to Y.

Definition 2.1.59. *The sequence f_n **converges pointwise** to $f \colon X \to Y$ if*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all $x \in X$. Using mathematical symbols, we can write

$$\forall x \in X,\, \forall \epsilon > 0,\, \exists N \in \mathbb{N},\, \forall n \in \{n' \in \mathbb{N} \,|\, n' > N\},\, d_Y\left(f_n(x), f(x)\right) < \epsilon.$$

Definition 2.1.60. *The sequence f_n **converges uniformly** to $f \colon X \to Y$ if*

$$\forall \epsilon > 0,\, \exists N \in \mathbb{N},\, \forall n \in \{n' \in \mathbb{N} \,|\, n' > N\},\, \forall x \in X,\, d_Y\left(f_n(x), f(x)\right) < \epsilon.$$

This condition is also equivalent to

$$\lim_{n\to\infty} \sup_{x\in X} d_Y\left(f_n(x), f(x)\right) = 0.$$

Theorem 2.1.61. *If each f_n is continuous and f_n converges uniformly to $f\colon X \to Y$, then f is continuous.*

Proof. The goal is to show that, for all $x \in X$ and any $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), f(y)) < \epsilon$ if $d_X(x, y) < \delta$. Since $f_n \to f$ uniformly, for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $d_Y(f_n(x), f(x)) < \epsilon/3$ for all $n > N$ and all $x \in X$. Now, we can fix $\epsilon > 0$ use the N promised above. Then, for any $n > N$, the continuity of f_n implies that, for all $x \in X$ and any $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f_n(x), f_n(y)) < \epsilon/3$ if $d_X(x, y) < \delta$. Thus, if $d_X(x, y) < \delta$, then

$$\begin{aligned} d_Y(f(x), f(y)) &\leq d_Y(f(x), f_n(x)) + d_Y(f_n(x), f_n(y)) + d_Y(f_n(y), f(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

□

2.2 General Topology*

While topology originated with the study of sets of finite-dimensional real vectors, its mathematical abstraction can also be useful. We note that some of the terms used above, for metric spaces, are redefined below. Fortunately, these new definitions are compatible with the old ones when the topology is generated by a metric.

Definition 2.2.1. *A **topology** on a set X is a collection $\mathcal{J}$ of subsets of X that satisfies the following properties,*

1. *$\emptyset$ and X are in $\mathcal{J}$*
2. *the union of the elements of any subcollection of $\mathcal{J}$ is in $\mathcal{J}$*
3. *the intersection of the elements of any finite subcollection of $\mathcal{J}$ is in $\mathcal{J}$.*

A subset $A \subseteq X$ is called an **open set** of X if $A \in \mathcal{J}$. Using this terminology, a topological space is a set X together with a collection of subsets of X, called *open sets*, such that $\emptyset$ and X are both open and such that arbitrary unions and finite intersections of open sets are open.

Definition 2.2.2. *If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called* basis elements*) such that:*

1. *for each $x \in X$, there exists a basis element B containing x.*

2. *if $x \in B_1$ and $x \in B_2$ where $B_1, B_2 \in \mathcal{B}$, then there exists a basis element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.*

3. *a subset $A \subseteq X$ is open in the topology on X generated by $\mathcal{B}$ if and only if, for every $x \in A$, there exists a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq A$.*

Probably the most important and frequently used way of imposing a topology on a set is to define the topology in terms of a metric.

Example 2.2.3. *If d is a metric on the set X, then the collection of all ϵ-balls*

$$\{B_d(x, \epsilon) | x \in X, \epsilon > 0\}$$

is a basis for a topology on X. This topology is called the ***metric topology*** *induced by d.*

Applying the meaning of open set from Definition 2.2.2 to this basis, one finds that a set A is open if and only if, for each $x \in A$, there exists a $\delta > 0$ such that $B_d(x, \delta) \subset A$. Clearly, this condition agrees with the definition of d-open from Definition 2.1.13.

Definition 2.2.4. *Let X be a topological space. This space is said to be* ***metrizable*** *if there exists a metric d on the set X that induces the topology of X.*

We note that definitions and results in Sections 2.1.3 and 2.1.4 for metric spaces actually apply to any metrizable space. For example, a metrizable space is complete if and only if there the metric that induces its topology also defines a complete metric space.

Example 2.2.5. *While most of the spaces discussed in these notes are metrizable, there is a very common notion of convergence that is not metrizable. The topology on the set of functions $f \colon [0, 1] \to \mathbb{R}$ where the open sets are defined by pointwise convergence is not metrizable.*

2.2.1 Closed Sets and Limit Points

Definition 2.2.6. *A subset A of a topological space X is **closed** if the set*

$$A^c = X - A = \{x \in X | x \notin A\}$$

is open.

Note that a set can be open, closed, both, or neither! It can be shown that the collection of closed subsets of a space X has properties similar to those satisfied by the collection of open subsets of X.

Fact 2.2.7. *Let X be a topological space. The following conditions hold,*

1. *$\emptyset$ and X are closed*
2. *arbitrary intersections of closed sets are closed*
3. *finite unions of closed sets are closed.*

Definition 2.2.8. *Given a subset A of a topological space X, the **interior** of A is defined as the union of all open sets contained in A. The **closure** of A is defined as the intersection of all closed sets containing A.*

The interior of A is denoted by A° and the closure of A is denoted by $\overline{A}$. We note that A° is open and $\overline{A}$ is closed. Furthermore, $A^\circ \subseteq A \subseteq \overline{A}$.

Theorem 2.2.9. *Let A be a subset of the topological space X. The element x is in $\overline{A}$ if and only if every open set B containing x intersects A.*

Proof. We prove instead the equivalent contrapositive statement: $x \notin \overline{A}$ if and only if there is an open set B containing x that does not intersect A. Clearly, if $x \notin \overline{A}$, then $\overline{A}^c = X - \overline{A}$ is an open set containing x that does not intersect A. Conversely, if there is an open set B containing x that does not intersect A, then $B^c = X - B$ is a closed set containing A. The definition of closure implies that B^c must also contain $\overline{A}$. But $x \notin B^c$, so $x \notin \overline{A}$. □

Definition 2.2.10. *An open set O containing x is called a **neighborhood** of x.*

Definition 2.2.11. *Suppose A is a subset of the topological space X and let x be an element of X. Then x is a **limit point** of A if every neighborhood of x intersects A in some point other than x itself.*

In other words, $x \in X$ is a limit point of $A \subset X$ if $x \in \overline{A - \{x\}}$, the closure of $A - \{x\}$. The point x may or may not be in A.

Theorem 2.2.12. *A subset of a topological space is closed if and only if it contains all its limit points.*

Definition 2.2.13. *A subset A of a topological space X is **dense** in X if every $x \in X$ is a limit point of the set A. This is equivalent to its closure $\overline{A}$ being equal to X.*

Definition 2.2.14. *A topological space X is **separable** if it contains a countable subset that is dense in X.*

Example 2.2.15. *Since every real number is a limit point of rational numbers, it follows that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. This also implies that $\mathbb{R}$, the standard metric space of real numbers, is separable.*

2.2.2 Continuity

Definition 2.2.16. *Let X and Y be topological spaces. A function $f \colon X \to Y$ is **continuous** if for each open subset $O \subseteq Y$, the set $f^{-1}(O)$ is an open subset of X.*

Recall that $f^{-1}(B)$ is the set $\{x \in X | f(x) \in B\}$. Continuity of a function depends not only upon the function f itself, but also on the topologies specified for its domain and range!

Theorem 2.2.17. *Let X and Y be topological spaces and consider a function $f \colon X \to Y$. The following are equivalent:*

1. *f is continuous*
2. *for every subset $A \subseteq X$, $f\left(\overline{A}\right) \subseteq \overline{f(A)}$*
3. *for every closed set $C \subseteq Y$, the set $f^{-1}(C)$ is closed in X.*

Proof. $(1 \Rightarrow 2)$. Assume f is a continuous function. We wish to show $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ for every subset $A \subseteq X$. To begin, suppose A is fixed and let $y \in f\left(\overline{A}\right)$. Then, there exists $x \in \overline{A}$ such that $f(x) = y$. Let $O \subseteq Y$ be a neighborhood of $f(x)$. Preimage $f^{-1}(O)$ is an open set containing x because f is continuous. Since $x \in \overline{A} \cap f^{-1}(O)$, we gather that $f^{-1}(O)$ must intersect with A in some point

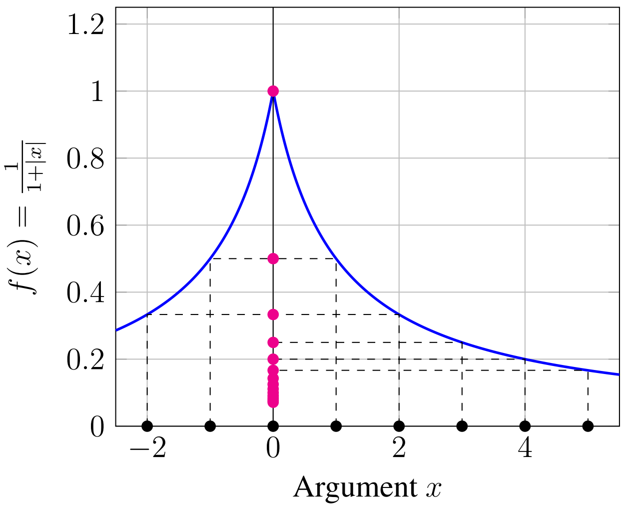

Figure 2.3: The function $f(x) = \frac{1}{1+|x|}$ is continuous. The set of integers $\mathbb{Z}$ is closed. Yet, the image of this set, $f(\mathbb{Z}) = \{1/n : n \in \mathbb{N}\}$, is not closed. Thus, this is an example of a continuous function along with a set for which $f\left(\overline{\mathbb{Z}}\right) \subsetneq \overline{f\left(\mathbb{Z}\right)}$.

x'. Moreover, $f(x') \in f\left(f^{-1}(O)\right) \subseteq O$ and $f(x') \in f(A)$. Thus, O intersects with $f(A)$ in the point $f(x')$. Since O is an arbitrary neighborhood of $f(x)$, we deduce that $f(x) \in \overline{f(A)}$ by Theorem 2.2.9. Collecting these results, we get that any $y \in f\left(\overline{A}\right)$ is also in $\overline{f(A)}$.

$(2 \Rightarrow 3)$. For this step, we assume that $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ for every subset $A \subseteq X$. Let $C \subseteq Y$ be a closed set and let $A = f^{-1}(C)$. Then, $f(A) = f\left(f^{-1}(C)\right) \subseteq C$. If $x \in \overline{A}$, we get

$$f(x) \in f\left(\overline{A}\right) \subseteq \overline{f(A)} \subseteq \overline{C} = C.$$

So that $x \in f^{-1}(C) = A$ and, as a consequence, $\overline{A} \subseteq A$. Thus, $A = \overline{A}$ is closed.

$(3 \Rightarrow 1)$. Let O be an open set in Y. Let $O^c = Y - O$; then O^c is closed in Y. By assumption, $f^{-1}(O^c)$ is closed in X. Using elementary set theory, we have

$$X - f^{-1}(O^c) = \{x \in X | f(x) \notin O^c\} = \{x \in X | f(x) \in O\} = f^{-1}(O).$$

That is, $f^{-1}(O)$ is open. $\square$

Theorem 2.2.18. *Suppose X and Y are two metrizable spaces with metrics d_X and d_Y. Consider a function $f \colon X \to Y$. The function f is continuous if and only if it is d-continuous with respect to these metrics.*

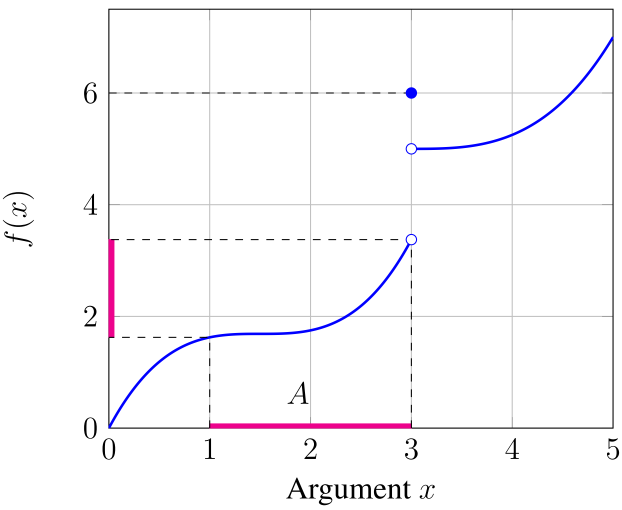

Figure 2.4: Given a function with a discontinuity and a set A, the image of the closure, $f\left(\overline{A}\right)$, need not be a subset of the closure of the image, $\overline{f(A)}$, as seen in the example above.

Proof. Suppose that f is continuous. For any $x_1 \in X$ and $\epsilon > 0$, let $O_y = B_{d_Y}(f(x_1), \epsilon)$ and consider the set

$$O_x = f^{-1}\left(O_y\right)$$

which is open in X and contains the point x_1. Since O_x is open and $x_1 \in O_x$, there exists a d-open ball $B_{d_X}(x_1, \delta)$ of radius $\delta > 0$ centered at x_1 such that $B_{d_X}(x_1, \delta) \subset O_x$. We also see that $f(x_2) \in O_y$ for any $x_2 \in B_{d_X}(x_1, \delta)$ because $A \subseteq O_x$ implies $f(A) \subseteq O_y$. It follows that $d_Y\left(f(x_1), f(x_2)\right) < \epsilon$ for all $x_2 \in B_{d_X}(x_1, \delta)$.

Conversely, let O_y be an open set in Y and suppose that the function f is d-continuous with respect to d_X and d_Y. For any $x \in f^{-1}(O_y)$, there exists a d-open ball $B_{d_Y}(f(x), \epsilon)$ of radius $\epsilon > 0$ centered at $f(x)$ that is entirely contained in O_y. By the definition of d-continuous, there exits a d-open ball $B_{d_X}(x, \delta)$ of radius $\delta > 0$ centered at x such that $f\left(B_{d_X}(x, \delta)\right) \subset B_{d_Y}\left(f(x), \epsilon\right)$. Therefore, every $x \in f^{-1}(O_y)$ has a neighborhood in the same set, and that implies $f^{-1}(O_y)$ is open. $\square$

Definition 2.2.19. *A sequence $x_1, x_2, \ldots$ of points in X is said to* ***converge*** *to $x \in X$ if for every neighborhood O of x there exists a positive integer N such that $x_i \in O$ for all $i \geq N$.*

A sequence need not converge at all. However, if it converges in a metrizable space, then it converges to only one element.

Theorem 2.2.20. *Suppose that X is a metrizable space, and let $A \subseteq X$. There exists a sequence of points of A converging to x if and only if $x \in \overline{A}$.*

Proof. Suppose $x_n \to x$, where $x_n \in A$. Then, for every open set O containing x, there is an N, such that $x_n \in O$ for all $n > N$. By Theorem 2.2.9, this implies that $x \in \overline{A}$. Let d be a metric for the topology of X and x be a point in $\overline{A}$. For each positive integer n, consider the neighborhood $B_d\left(x, \frac{1}{n}\right)$. Since $x \in \overline{A}$, the set $A \cap B_d\left(x, \frac{1}{n}\right)$ is not empty and we choose x_n to be any point in this set. It follows that the sequence $x_1, x_2, \ldots$ converges to x. Notice that the "only if" proof holds for any topological space, while "if" requires a metric. □

Theorem 2.2.21. *Let $f \colon X \to Y$ where X is a metrizable space. The function f is continuous if and only if for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$.*

Proof. Suppose that f is continuous. Let O be a neighborhood of $f(x)$. Then $f^{-1}(O)$ is a neighborhood of x, and so there exists an integer N such that $x_n \in f^{-1}(O)$ for $n \geq N$. Thus, $f(x_n) \in O$ for all $n \geq N$ and $f(x_n) \to f(x)$.

To prove the converse, assume that the convergent sequence condition is true. Let $A \subseteq X$. Since X is metrizable, one finds that $x \in \overline{A}$ implies that there exists a sequence $x_1, x_2, \ldots$ of points of A converging to x. By assumption, $f(x_n) \to f(x)$. Since $f(x_n) \in f(A)$, Theorem 2.2.17 implies that $f(x) \in \overline{f(A)}$. Hence $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ and f is continuous. □