ECE 586: Vector Space Methods
Course Introduction Video

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Overview

- Engineering uses science and mathematics to invent, design, and build things that solve problems and achieve practical goals
- From a mathematical perspective, engineering models the world using vectors and analyzes it using logic, linear algebra, and optimization
- This class is essentially an applied math class designed for graduate engineers in data science, signal processing, and quantum computing
- Its goal is to refresh and extend your undergraduate knowledge of linear systems, vector spaces, matrices, and optimization

Major Topics

- Logic and Set Theory
- Metric Spaces and Topology
  - Practical applications include the analysis of iterative algorithms
- Linear Algebra: Normed and Inner-Product Spaces
  - Practical applications include spaces of functions and Markov chains
- Approximation and Projection
  - Practical applications include function approximation and machine learning
- The Four Fundamental Subspaces and Singular Value Decomposition
  - Practical applications include dimensionality reduction
- Optimization in Vector Spaces
  - Practical applications include duality bounds in convex optimization

Example Problem 1

- Approximation Error in Linear Systems
  - Suppose you know a vector b satisfies $Ax = b$ for an invertible matrix A
  - To compute x, you could multiply by A^{-1} to get $A^{-1}b = A^{-1}(Ax) = x$
  - But, what if A is not known perfectly?
  - Let's assume we know $\hat{A} = A + E$ where the error matrix E is "small"
  - Now, we can compute $\hat{x} = \hat{A}^{-1}b$. But, what can we say about $x - \hat{x}$?
- The mathematical study of "closeness" is called Topology
  - To answer the above question, one can define distances between objects
  - Using these distances, we can define a constant $c(\hat{A})$ such that

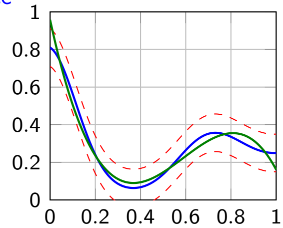
$$\text{distance}(x, \hat{x}) \leq c(\hat{A})\ \text{distance}(A, \hat{A}) = c(\hat{A})\ \text{length}(E)$$

Example Problem 2

- Approximation of Functions by Simpler Functions
  - Suppose you want to find a 3rd-order polynomial that gives a "good" approximation of a function $f(x)$ on the interval $[0,1]$.
  - A key question is, "What do we mean by good?" max error? avg error?
  - A standard approach is to minimize the mean squared error and this gives

$$\min_{a_0,a_1,a_2,a_3} \int_0^1 \left[f(x) - (a_0 + a_1 x + a_2 x^2 + a_3 x^3) \right]^2 dx$$

  - Given $f(x)$ and the ability to integrate, one can solve this via brute force
  - This course will show there is also an simple intuitive approach



- Solution via Orthogonality and Linear Algebra
  - Compute a_0, a_1, a_2, a_3 using 4 linear equations ($k = 0, 1, 2, 3$) given by

$$\int_0^1 (a_0 + a_1 x + a_2 x^2 + a_3 x^3) x^k dx = \int_0^1 f(x) x^k dx.$$

Example Problem 3

- Convex Optimization
  - Consider a real function $f(x)$ on the real interval $[a, b]$
  - A chord of f is a line connecting $(x, f(x))$ and $(y, f(y))$
  - $f(x)$ is convex if all chords lie above the function, i.e. for $\lambda \in [0, 1]$,

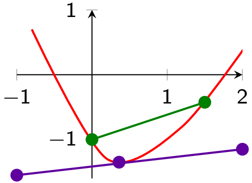
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

  - Convexity implies all local minima have the same minimum value

Q: Can we compute a simple lower bound on the minimum value?

- Solution via Tangent Line
  - Yes, a convex function also lies above any tangent line and

$$\min_{x\in[a,b]} f(x) \geq \min_{x\in[a,b]} \left[f(x_0) + (x - x_0)f'(x_0)\right]$$



Course Overview and Philosophy

- First Theme: Standard definitions and vocabulary
  - Engineering has many subfields where new ideas are described using abstract mathematics
  - A key goal of this course is to teach the vocabulary necessary to read and understand papers and textbooks in these subfields
- Second Theme: Standard theorems and their proof
  - Our focus is on mathematical foundations of engineering analysis
  - Proofs illustrate mathematical reasoning and show why results are true
  - Emphasis is on general techniques that extend to advanced problems
- Assignments
  - Written homework builds knowledge of definitions and techniques
  - Computer assignments demonstrate practical applications
  - These are like training for a sport; each workout provides a small gain but consistent effort leads to significant improvement over time

Learning Strategies

- Course Elements
  - Syllabus: Lists lecture/video topics by date along with HW
  - Course Notes: Detailed description of the course content
  - Flip Videos: Slide-based introduction to course notes ☺
  - Lectures: Q/A about videos/notes + group work on example problems
- My Recommendations
  - Watch the flip video first and then read the matching course notes
  - Afterward, try the homework while referring to the course notes as needed
  - If you're stuck $>$30min on a question, ask for a hint (via e-mail or web)
  - Visit office hours to get additional help as needed
  - Make sure to watch the video before the class meeting!