ECE 586: Vector Space Methods
Lecture 10 Flip Video: Vector Spaces

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Definition: A vector space consists of the following,

1. a **field** $\boldsymbol{F}$ of scalars (where $0, 1 \in F$ denote the add. and mult. identities)
2. a **set** $\boldsymbol{V}$ of vectors (which are decorated by an underline in these notes)
3. a **binary operation called vector addition**, which maps any pair of vectors $\underline{v}, \underline{w} \in V$ to a vector $\underline{v} + \underline{w} \in V$ satisfying four conditions:
    1. vector addition is commutative: $\underline{v} + \underline{w} = \underline{w} + \underline{v}$
    2. vector addition is associative: $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$
    3. there is a unique vector $\underline{0} \in V$ such that $\underline{v} + \underline{0} = \underline{v}$, $\forall \underline{v} \in V$
    4. to each $\underline{v} \in V$ there is a unique vector $-\underline{v} \in V$ such that $\underline{v} + (-\underline{v}) = \underline{0}$
4. a **binary operation called scalar multiplication**, which maps any $s \in F$ and $\underline{v} \in V$ to a vector $s\underline{v} \in V$ satisfying four conditions:
    1. the identity is multiplicative identity of the field: $1\underline{v} = \underline{v}$, $\forall \underline{v} \in V$
    2. scalar multiplication is associative: $(s_1 s_2)\underline{v} = s_1(s_2\underline{v})$
    3. scalar multiplication distributes over vector addition: $s(\underline{v} + \underline{w}) = s\underline{v} + s\underline{w}$
    4. scalar addition distributes scalar multiplication: $(s_1 + s_2)\underline{v} = s_1\underline{v} + s_2\underline{v}$.

Example (Standard vector space for F^n)

Let F be a field, and let $V = F^n$ be the set of n-tuples $\underline{v} = (v_1, \ldots, v_n)$. For $\underline{w} = (w_1, \ldots, w_n) \in F^n$, the sum of $\underline{v}$ and $\underline{w}$ is defined by

$$\underline{v} + \underline{w} = (v_1 + w_1, \ldots, v_n + w_n).$$

The scalar product of $s \in F$ and $\underline{v} \in V$ is defined $s\underline{v} = (sv_1, \ldots, sv_n)$.

Example (General vector space of functions)

Let X be a non-empty set and let Y be a vector space over F. Consider the set V of all functions mapping X to Y. The vector addition of two functions $f, g \in V$ is the function $(f + g)\colon X \to Y$ defined by

$$(f + g)(x) = f(x) + g(x) \quad \forall x \in X,$$

where the RHS is defined by vector addition in Y. The scalar product of $s \in F$ and the function $f \in V$ is the function sf defined by $(sf)(x) = s\,f(x)$ for all $x \in X$, where the RHS is defined by scalar multiplication in Y.

3.3.1: Subspaces

Definition

Let V be a vector space over F. A subspace of V is a subset $W \subset V$ which is itself a vector space over F.

Lemma

A non-empty subset $W \subset V$ is a subspace of V if and only if, for every pair $\underline{w}_1, \underline{w}_2 \in W$ and every scalar $s \in F$, the vector $s\underline{w}_1 + \underline{w}_2 \in W$.

Sketch proof in live session (via inheritance from V).

Example

Let A be an $m \times n$ matrix over F. Then, the subset $V \subseteq F^{n\times 1}$ of vectors satisfying $A\underline{v} = \underline{0}$ forms a subspace.

3.3: Linear Combinations

Definition

A vector $\underline{w} \in V$ is said to be a linear combination of the vectors $\underline{v}_1, \ldots, \underline{v}_n \in V$ provided that there exist scalars $s_1, \ldots, s_n \in F$ such that

$$\underline{w} = \sum_{i=1}^{n} s_i \underline{v}_i .$$

Definition

Let U be a list (or set) of vectors in V. The span of U, denoted $\mathrm{span}(U)$, is defined to be the set of all finite linear combinations of vectors in U.

Example

For a vector space V, the span of any list (or set) of vectors in V forms a subspace.

Definition

Let V be a vector space over F. A list of vectors $\underline{u}_1, \ldots, \underline{u}_n \in V$ is called linearly dependent if there are scalars $s_1, \ldots, s_n \in F$, not all equal to 0, such that

$$\sum_{i=1}^{n} s_i \underline{u}_i = \underline{0}.$$

A list that is not linearly dependent is called linearly independent.

Similarly, a subset $U \subset V$ is called linearly dependent if there is a finite list $\underline{u}_1, \ldots, \underline{u}_n \in U$ of distinct vectors that is linearly dependent. Otherwise, it is called linearly independent.

Example

For $V = \mathbb{R}^4$, vectors $\underline{v}_1 = (1, 1, 0, 0)$, $\underline{v}_2 = (0, 1, 1, 0)$, $\underline{v}_3 = (0, 0, 1, 1)$ are linearly independent because $\underline{u} = s_1\underline{v}_1 + s_2\underline{v}_2 + s_3\underline{v}_3 \neq \underline{0}$ unless $s_1 = s_2 = s_3 = 0$ (e.g., $u_1 \neq 0$ if $s_1 \neq 0$, $u_4 \neq 0$ if $s_3 \neq 0$, and $u_2 \neq 0$ if $s_2 \neq 0 \wedge s_1 = 0$).

3.3.2: Basis

Definition

Let V be a vector space over F. Let $\mathcal{B} = \{\underline{v}_\alpha | \alpha \in A\}$ be a subset of linearly independent vectors from V such that every $\underline{v} \in V$ can be written as a finite linear combination of vectors from $\mathcal{B}$. Then, the set $\mathcal{B}$ is a Hamel basis for V. If V has a finite basis, it is called finite-dimensional.

From this, a basis decomposition $\underline{v} = \sum_{i=1}^{n} s_i \underline{v}_{\alpha_i}$ must be unique:

The difference between any two distinct decompositions produces a finite linear dependency in the basis and, hence, a contradiction.

Theorem

Every vector space has a Hamel basis.

Example

Let F be a field and let $U \subset F^n$ be the list of vectors $\underline{e}_1, \dots, \underline{e}_n$ defined by

$$\begin{aligned} \underline{e}_1 &= (1, 0, \dots, 0) \\ \underline{e}_2 &= (0, 1, \dots, 0) \\ \vdots &= \qquad \vdots \\ \underline{e}_n &= (0, 0, \dots, 1). \end{aligned}$$

For any $\underline{v} = (v_1, \dots, v_n) \in F^n$, we have

$$\underline{v} = \sum_{i=1}^{n} v_i \underline{e}_i. \tag{1}$$

Thus, the collection $U = \{\underline{e}_1, \dots, \underline{e}_n\}$ spans F^n. Since $\underline{v} = \underline{0}$ in (1) if and only if $v_1 = \cdots = v_n = 0$, U is linearly independent. Accordingly, the set U is a basis for F^n. This basis is termed the standard basis of F^n.

3.3.2: Dimension

Theorem

Let V be a finite-dimensional vector space that is spanned by a finite set of vectors $W = \{\underline{w}_1, \ldots, \underline{w}_n\}$. If $U = \{\underline{u}_1, \ldots, \underline{u}_m\} \subset V$ is a linearly independent set of vectors, then $m \leq n$.

Proof in live session.

Thus, all bases have the same number of vectors.

Definition

The dimension of a finite-dimensional vector space is the number of elements in any basis for V. It is denoted by $\dim(V)$.

3.3.2: Invertibility

Lemma

Let $A \in F^{n \times n}$ be an invertible matrix. Then, the columns of A form a basis for F^n. Similarly, the rows of A will also form a basis for F^n.

Proof in live session.

Theorem

Let A be an $n \times n$ matrix over F whose columns, denoted by $\underline{a}_1, \ldots, \underline{a}_n$, form a linearly independent set of vectors in F^n. Then A is invertible.

Proof in live session.

Next Steps

- To continue studying after this video –
  - Try the required reading: Course Notes EF 3.1 - 3.3
  - Or the recommended reading: LADR Ch. 1, Ch. 2
  - Also, look at the problems in Assignment 4