ECE 586: Vector Space Methods
Lecture 11 Flip Video: Linear Transforms

Henry D. Pfister
Duke University

Sums and Direct Sums

Definition

Let U, W be subsets of a vector space V. The sum of U and W is defined by

$$U + W \triangleq \{\underline{v} \in V \,|\, \exists \underline{u} \in U, \exists \underline{w} \in W, \underline{v} = \underline{u} + \underline{w}\}\,.$$

Definition

For a vector space V, subspaces U and W are called disjoint if $U \cap W = \{\underline{0}\}$.

Definition

For disjoint subspaces U and W in a vector space, their direct sum equals their sum but is denoted by $U \oplus W$ to emphasize that U and W are disjoint.

An important property of a direct sum is that any vector $\underline{v} \in U \oplus W$ has a unique decomposition $\underline{v} = \underline{u} + \underline{w}$ where $\underline{u} \in U$ and $\underline{w} \in W$.

3.3.3 Coordinate Systems and Vectors

Definition

If V is a finite-dimensional vector space, an ordered basis for V is a finite list of vectors that is linearly independent and spans V.

Remark

If $\mathcal{B} = (\underline{v}_1, \ldots, \underline{v}_n)$ is an ordered basis for V, then the set $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is a basis for V. But, $\mathcal{B}$ defines the set and a specific ordering for the vectors.

Definition

For a finite-dimensional vector space V with ordered basis $\mathcal{B} = (\underline{v}_1, \ldots, \underline{v}_n)$, the coordinate vector of $\underline{v} \in V$ is denoted by $[\underline{v}]_{\mathcal{B}}$ and equals the unique vector $\underline{s} = F^n$ such that

$$\underline{v} = \sum_{i=1}^{n} s_i \underline{v}_i .$$

Try computing $[(1, 2, 3, 4)]_{\mathcal{B}}$ for $\mathcal{B} = ((1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1))$.

3.4: Linear Transforms

Definition

Let V and W be vector spaces over a field F. A linear transform from V to W is a function T from V into W such that

$$T\left(s\underline{v}_1 + \underline{v}_2\right) = sT\underline{v}_1 + T\underline{v}_2$$

for all vectors $\underline{v}_1, \underline{v}_2 \in V$ and all scalars $s \in F$ (i.e., T is linear).

Example

Let A be a fixed $m \times n$ matrix over F. The function T defined by $T\left(\underline{v}\right) = A\underline{v}$ is a linear transformation from $F^{n\times 1}$ into $F^{m\times 1}$.

Example

Let V be the space of continuous functions from $[0,1]$ to $\mathbb{R}$. Define T by

$$(Tf)(x) = \int_0^x f(t)dt.$$

Then, T is a linear transform from V to V because Tf is continuous.

3.4.2: Properties of Linear Transforms

Definition (Range)

For a linear transformation $T: V \to W$, the range of T is the subspace of vectors $\underline{w} \in W$ such that $\underline{w} = T\underline{v}$ for some $\underline{v} \in V$. It is denoted by

$$\mathcal{R}(T) \triangleq \{\underline{w} \in W | \exists \underline{v} \in V \text{ s.t. } T\underline{v} = \underline{w}\} = \{T\underline{v} | \underline{v} \in V\}.$$

Definition (Nullspace)

For a linear transformation $T: V \to W$, the nullspace of T is the subspace of vectors $\underline{v} \in V$ such that $T\underline{v} = \underline{0}$. We denote the nullspace of T by

$$\mathcal{N}(T) \triangleq \{\underline{v} \in V | T\underline{v} = \underline{0}\}.$$

Theorem

Let V, W be vector spaces over F and $\mathcal{B} = \{\underline{v}_\alpha | \alpha \in A\}$ be a Hamel basis for V. For each mapping $G: \mathcal{B} \to W$, there is a unique linear transformation $T: V \to W$ such that $T\underline{v}_\alpha = G(\underline{v}_\alpha)$ for all $\alpha \in A$.

Proof in live session.

3.4.2: Rank and Nullity

Definition (Rank and Nullity)

Let V and W be vector spaces over a field F, and let T be a linear transformation from V into W. The rank of T is the dimension of the range of T and the nullity of T is the dimension of the nullspace of T.

Theorem (Rank-Nullity)

Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. If V is finite-dimensional, then

$$\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(V)$$

Proof in live session.

Theorem

If A is an $m \times n$ matrix with entries in the field F, then

$$\text{row rank}(A) \triangleq \dim(\mathcal{R}(A^T)) = \dim(\mathcal{R}(A)) \triangleq \mathrm{rank}(A).$$

Proof in live session.

Next Steps

- To continue studying after this video –
  - Try the required reading: Course Notes EF 3.3 - 3.4
  - Or the recommended reading: LADR Ch. 3ABC
  - Also, look at the problems in Assignment 4