3.5: Normed Vector Spaces

Let $V$ be a vector space over the real numbers or the complex numbers.

**Definition**

A **norm** on vector space $V$ is a real-valued function $\|\cdot\| : V \to \mathbb{R}$ that satisfies the following properties.

1. $\|v\| \geq 0 \quad \forall v \in V$; equality holds if and only if $v = 0$
2. $\|sv\| = |s|\|v\| \quad \forall v \in V, s \in F$
3. $\|v + w\| \leq \|v\| + \|w\| \quad \forall v, w \in V$.

The concept of a norm is closely related to that of a metric. For instance, a metric can be defined from any norm.

Let $\|v\|$ be a norm on vector space $V$, then the **induced metric** is

$$d(v, w) = \|v - w\|.$$
3.5: Examples of Normed Vector Spaces

Example (Standard Norms for Real/Complex Vector Spaces)

The following functions are examples of norms for $\mathbb{R}^n$ and $\mathbb{C}^n$:

1. The $l^1$ norm: $\|v\|_1 = \sum_{i=1}^{n} |v_i|$

2. The $l^p$ norm: $\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}}$, $p \in (1, \infty)$

3. The $l^\infty$ norm: $\|v\|_\infty = \max_{1,\ldots,n} \{|v_i|\}$

Example (Standard Norms for Real/Complex Function Spaces)

Similarly, for the vector space of functions from $[a, b]$ to $\mathbb{R}$ (or $\mathbb{C}$):

1. The $L^1$ norm: $\|f(t)\|_1 = \int_{a}^{b} |f(t)| \, dt$

2. The $L^p$ norm: $\|f(t)\|_p = \left(\int_{a}^{b} |f(t)|^p \, dt\right)^{\frac{1}{p}}$, $p \in (1, \infty)$

3. The $L^\infty$ norm: $\|f(t)\|_\infty = \text{ess sup}_{[a,b]} \{|f(t)|\}$

For infinite dimensional spaces, only vectors with finite norm are included.
3.5: Norms Versus Metrics

Example

Consider vectors in $\mathbb{R}^n$ with the euclidean metric

$$d(v, w) = \sqrt{(v_1 - w_1)^2 + \cdots + (v_n - w_n)^2}.$$

Recall the bounded metric given by

$$\tilde{d}(v, w) = \min \{d(v, w), 1\}.$$

Define $f: \mathbb{R}^n \rightarrow \mathbb{R}$ by $f(v) = \tilde{d}(v, 0)$. Is the function $f$ a norm?

By the properties of $\tilde{d}$, we have

1. $\tilde{d}(v, 0) \geq 0 \quad \forall v \in V$; equality holds if and only if $v = 0$
2. $\tilde{d}(v + w, 0) = \tilde{d}(v, -w) \leq \tilde{d}(v, 0) + \tilde{d}(w, 0) \quad \forall v, w \in V$.

However, $\tilde{d}(sv, 0)$ is not always equal to $s\tilde{d}(v, 0)$. For instance, $\tilde{d}(2e_1, 0) = 1 < 2\tilde{d}(e_1, 0)$. Thus, the $f(v) = \tilde{d}(v, 0)$ is not a norm.
3.5: Complete Normed Spaces

**Definition**

A vector \( v \in V \) is called **normalized** if \( \|v\| = 1 \). For any \( v \neq 0 \), consider

\[
u = v / \|v\|
\]

with norm \( \|u\| = 1 \). A normalized vector is called a **unit vector**.

**Definition**

A complete normed vector space is called a **Banach space**.

**Example**

Vector spaces \( \mathbb{R}^n \) (or \( \mathbb{C}^n \)) with any well-defined norm are Banach spaces.

**Example**

The vector space of continuous functions \( f : [a, b] \to \mathbb{R} \) is a Banach space under the norm

\[
\|f(t)\| = \sup_{t \in [a,b]} f(t).
\]
3.5: Schauder Basis

Definition
A Banach space $V$ has a **Schauder basis**, $v_1, v_2, \ldots$, if every $v \in V$ can be written uniquely as

$$v = \sum_{i=1}^{\infty} s_i v_i,$$

where convergence is determined by the norm topology.

Example
Let $V = \mathbb{R}^\infty$ be the vector space of semi-infinite real sequences. The **standard Schauder basis** is the countably infinite extension $\{e_1, e_2, \ldots\}$ of the standard basis.
3.5: Convergence of Sums

Banach space convergence via the induced metric $d(v, w) = \|v - w\|.$

**Lemma**

If $\sum_{i=1}^{\infty} \|v_i\| = a < \infty$, then $u_n = \sum_{i=1}^{n} v_i$ satisfies $u_n \to u$ with $\|u\| \leq a$.

**Proof.**

- Let $a_n = \sum_{i=1}^{n} \|v_i\|$ and observe that, for $n > m$,
  
  $$|a_n - a_m| = \left| \sum_{i=1}^{n} \|v_i\| - \sum_{i=1}^{m} \|v_i\| \right| = \sum_{i=m+1}^{n} \|v_i\|$$

  $$\|u_n - u_m\| = \left\| \sum_{i=1}^{n} v_i - \sum_{i=1}^{m} v_i \right\| = \left\| \sum_{i=m+1}^{n} v_i \right\| \leq \sum_{i=m+1}^{n} \|v_i\|$$

- Since $\sum_{i=1}^{\infty} \|v_i\|$ converges in $\mathbb{R}$, $a_n$ must be a Cauchy sequence.
- Since $\|u_n - u_m\| \leq |a_n - a_m|$, $u_n$ is also a Cauchy sequence.
- Once $u_n$ converges, the norm bound given by the triangle inequality.
3.5: Open and Closed Subspaces

**Definition**

A **closed subspace** of a Banach space is a subspace that is a closed set in the topology generated by the norm.

**Theorem**

*All finite dimensional subspaces of a Banach space are closed.*

**Example**

Let $W = \{w_1, w_2, \ldots\}$ be a linearly independent sequence of normalized vectors in a Banach space. The span of $W$ only includes finite linear combinations. However, a sequence of finite linear combinations, like

$$u_n = \sum_{i=1}^{n} \frac{1}{i^2} w_i,$$

converges to $\lim_{n \to \infty} u_n$ if it exists. Thus, the span of $W$ is not closed.

Show convergence in live session.
Next Steps

To continue studying after this video –

- Try the required reading: Course Notes EF 3.5
- Or the recommended reading: LADR Ch. 6A
- Also, look at the problems in Assignment 5