ECE 586: Vector Space Methods
Lecture 13 Flip Video: Normed Vector Spaces

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3.5: Normed Vector Spaces

Let V be a vector space over the real numbers or the complex numbers.

Definition

A norm on vector space V is a real-valued function $\|\cdot\| : V \to \mathbb{R}$ that satisfies the following properties.

1. $\|\underline{v}\| \geq 0 \quad \forall \underline{v} \in V$; equality holds if and only if $\underline{v} = \underline{0}$
2. $\|s\underline{v}\| = |s| \, \|\underline{v}\| \qquad \forall \underline{v} \in V, s \in F$
3. $\|\underline{v} + \underline{w}\| \leq \|\underline{v}\| + \|\underline{w}\| \qquad \forall \underline{v}, \underline{w} \in V.$

The concept of a norm is closely related to that of a metric. For instance, a metric can be defined from any norm.

Let $\|\underline{v}\|$ be a norm on vector space V, then the induced metric is

$$d(\underline{v}, \underline{w}) = \|\underline{v} - \underline{w}\|.$$

3.5: Examples of Normed Vector Spaces

Example (Standard Norms for Real/Complex Vector Spaces)

The following functions are examples of norms for $\mathbb{R}^n$ and $\mathbb{C}^n$:

1. the l^1 norm: $\|\underline{v}\|_1 = \sum_{i=1}^n |v_i|$
2. the l^p norm: $\|\underline{v}\|_p = \left(\sum_{i=1}^n |v_i|^p\right)^{\frac{1}{p}}, \quad p \in (1, \infty)$
3. the l^∞ norm: $\|\underline{v}\|_\infty = \max_{1,\ldots,n}\{|v_i|\}$

Example (Standard Norms for Real/Complex Function Spaces)

Similarly, for the vector space of functions from $[a, b]$ to $\mathbb{R}$ (or $\mathbb{C}$):

1. the L^1 norm: $\|f(t)\|_1 = \int_a^b |f(t)| dt$
2. the L^p norm: $\|f(t)\|_p = \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}}, \quad p \in (1, \infty)$
3. the L^∞ norm: $\|f(t)\|_\infty = \operatorname{ess\,sup}_{[a,b]}\{|f(t)|\}$

For infinite dimensional spaces, only vectors with finite norm are included.

Example

Consider vectors in $\mathbb{R}^n$ with the euclidean metric

$$d\left(\underline{v}, \underline{w}\right) = \sqrt{(v_1 - w_1)^2 + \cdots + (v_n - w_n)^2}.$$

Recall the bounded metric given by

$$\bar{d}\left(\underline{v}, \underline{w}\right) = \min\left\{d\left(\underline{v}, \underline{w}\right), 1\right\}.$$

Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f\left(\underline{v}\right) = \bar{d}\left(\underline{v}, \underline{0}\right)$. Is the function f a norm?

By the properties of $\bar{d}$, we have

1. $\bar{d}\left(\underline{v}, \underline{0}\right) \geq 0 \quad \forall \underline{v} \in V$; equality holds if and only if $\underline{v} = \underline{0}$
2. $\bar{d}\left(\underline{v} + \underline{w}, \underline{0}\right) = \bar{d}\left(\underline{v}, -\underline{w}\right) \leq \bar{d}\left(\underline{v}, \underline{0}\right) + \bar{d}\left(\underline{w}, \underline{0}\right) \quad \forall \underline{v}, \underline{w} \in V$.

However, $\bar{d}\left(s\underline{v}, \underline{0}\right)$ is not always equal to $s\bar{d}\left(\underline{v}, \underline{0}\right)$. For instance, $\bar{d}\left(2\underline{e}_1, \underline{0}\right) = 1 < 2\bar{d}\left(\underline{e}_1, \underline{0}\right)$. Thus, the $f\left(\underline{v}\right) = \bar{d}\left(\underline{v}, \underline{0}\right)$ is not a norm.

3.5: Properies of Normed Vector Spaces

Definition

A vector $\underline{v} \in V$ is called normalized if $\|\underline{v}\| = 1$. For any $\underline{v} \neq \underline{0}$, consider

$$\underline{u} = \underline{v} \,/\, \|\underline{v}\|$$

with norm $\|\underline{u}\| = 1$. A normalized vector is called a unit vector.

Definition

For a vector space V over $\mathbb{R}$ or $\mathbb{C}$, two norms (denoted $\|\cdot\|_V$ and $\|\cdot\|_{V'}$) are called equivalent if, for all $\underline{v} \in V$, there are positive reals m, M such that

$$m\|\underline{v}\|_{V'} \leq \|\underline{v}\|_V \leq M\|\underline{v}\|_{V'}.$$

Lemma

For finite-dimensional normed spaces, all norms are equivalent.

Definition

A complete normed vector space is called a Banach space.

Example

Vector spaces $\mathbb{R}^n$ (or $\mathbb{C}^n$) with any well-defined norm are Banach spaces.

Example

Let $V = \mathbb{R}^\infty$ be the Banach space of real sequences $(v_1, v_2, v_3, \ldots)$ with norm

$$\|\underline{v}\|_p = \left(\sum_{i=1}^{\infty} |v_i|^p\right)^{1/p}.$$

Example

The vector space of continuous functions $f : [a, b] \to \mathbb{R}$ is a Banach space under the norm

$$\|f(t)\| = \sup_{t \in [a,b]} f(t).$$

3.5: Schauder Basis

Definition

A Banach space V has a Schauder basis, $\underline{v}_1, \underline{v}_2, \ldots$, if every $\underline{v} \in V$ can be written uniquely as

$$\underline{v} = \sum_{i=1}^{\infty} s_i \underline{v}_i,$$

where convergence is determined by the norm topology.

Example

Let $V = \mathbb{R}^\infty$ be the vector space of real sequences. The standard Schauder basis is the countably infinite extension $\{\underline{e}_1, \underline{e}_2, \ldots\}$ of the standard basis.

Caveat

Although all commonly used Banach spaces have a Schauder basis, some do not. Thus, this cannot be assumed in general proofs for Banach spaces.

3.5: Convergence of Sums

Banach space convergence via the induced metric $d(\underline{v}, \underline{w}) = \|\underline{v} - \underline{w}\|$.

Lemma

If $\sum_{i=1}^{\infty} \|\underline{v}_i\| = a < \infty$, *then* $\underline{u}_n = \sum_{i=1}^{n} \underline{v}_i$ *satisfies* $\underline{u}_n \to \underline{u}$ *with* $\|\underline{u}\| \leq a$.

Proof.

- Let $a_n = \sum_{i=1}^{n} \|\underline{v}_i\|$ and observe that, for $n > m$,

$$|a_n - a_m| = \left| \sum_{i=1}^{n} \|\underline{v}_i\| - \sum_{i=1}^{m} \|\underline{v}_i\| \right| = \sum_{i=m+1}^{n} \|\underline{v}_i\|$$

$$\|\underline{u}_n - \underline{u}_m\| = \left\| \sum_{i=1}^{n} \underline{v}_i - \sum_{i=1}^{m} \underline{v}_i \right\| = \left\| \sum_{i=m+1}^{n} \underline{v}_i \right\| \leq \sum_{i=m+1}^{n} \|\underline{v}_i\|$$

- Since $\sum_{i=1}^{\infty} \|\underline{v}_i\|$ converges in $\mathbb{R}$, a_n must be a Cauchy sequence
- Since $\|\underline{u}_n - \underline{u}_m\| \leq |a_n - a_m|$, $\underline{u}_n$ is also a Cauchy sequence
- Once $\underline{u}_n$ converges, the norm bound given by the triangle inequality $\square$

Definition

A closed subspace of a Banach space is a subspace that is a closed set in the topology generated by the norm.

Theorem

All finite dimensional subspaces of a Banach space are closed.

Example (Span of infinite linearly independent set not closed)

Let $W = \{\underline{w}_1, \underline{w}_2, \ldots\}$ be a linearly independent sequence of normalized vectors in a Banach space. The span of W only includes finite linear combinations. However, a sequence of finite linear combinations, like

$$\underline{u}_n = \sum_{i=1}^{n} \frac{1}{i^2} \underline{w}_i,$$

converges to $\lim_{n\to\infty} \underline{u}_n$ if it exists. Thus, the span of W is not closed.

Show convergence in live session.

Next Steps

- To continue studying after this video –
    - Try the required reading: Course Notes EF 3.5
    - Or the recommended reading: LADR Ch. 6A
    - Also, look at the problems in Assignment 5