6.1/6.3: Vector Space of Linear Transforms and Norms

**Definition**

Let $L(V, W)$ denote the vector space of all linear transforms from $V$ into $W$, where $V$ and $W$ are vector spaces over a field $F$.

An operator norm is a norm on a vector space of linear transforms.

**Definition (Induced Operator Norm)**

Let $V$ and $W$ be two normed vector spaces and let $T: V \to W$ be a linear transformation. The induced operator norm of $T$ is defined to

$$
\| T \|_{op} = \sup_{v \in V - \{0\}} \frac{\|Tv\|_W}{\|v\|_V} = \sup_{v \in V, \|v\|_V = 1} \|Tv\|_W.
$$

A common question about the operator norm is, “How do I know the two expressions give the same result?”. To see this, we can write

$$
\sup_{v \in V - \{0\}} \frac{\|Tv\|_W}{\|v\|_V} = \sup_{v \in V - \{0\}} \frac{\|Tv\|_W}{\|v\|_V} = \sup_{u \in V, \|u\|_V = 1} \|Tu\|_W.
$$
The induced operator norm has another property:

\[ \| T \| = \sup_{v \in V - \{0\}} \frac{\| T v \|}{\| v \|} \geq \frac{\| T u \|}{\| u \|}, \]

for non-zero \( u \in V \). This implies that \( \| T u \| \leq \| T \| \| u \| \) for all non-zero \( u \in V \). By checking \( u = 0 \) separately, one can see it holds for all \( u \in V \).

**Definition**

For the space \( L(V, V) \) of linear operators on \( V \), a norm is called submultiplicative if \( \| T U \| \leq \| T \| \| U \| \) for all \( T, U \in L(V, V) \).

Using the above inequality, induced operator norms are submultiplicative:

\[ \| U T v \| \leq \| U \| \| T v \| \leq \| U \| \| T \| \| v \|. \]
Continuity of Linear Transforms

Definition
A linear transform is called **bounded** if its induced operator norm is finite.

Theorem
A linear transformation $T : V \to W$ is bounded if and only if it is continuous.

Proof.
Suppose that $T$ is bounded; that is, there exists $M$ such that $\| Tv \| \leq M \| v \|$ for all $v \in V$. Let $v_1, v_2, \ldots$ be a convergent sequence in $V$, then

$$\| T v_i - T v_j \| = \| T (v_i - v_j) \| \leq M \| v_i - v_j \| .$$

This implies $T v_1, T v_2, \ldots$ is convergent in $W$ and, thus, $T$ is continuous.

Conversely, assume $T$ is continuous and notice that $T0 = 0$. Then, for any $\epsilon > 0$, there is a $\delta > 0$ such that $\| T v \| < \epsilon$ for all $\| v \| < \delta$. It follows that

$$\| Tu \| = \left\| T \frac{\delta u}{2 \| u \|} \right\| \frac{2 \| u \|}{\delta} < \frac{2\epsilon}{\delta} \| u \| .$$

Thus, $M = \frac{2\epsilon}{\delta}$ serves as an upper bound on $\| T \|$. □
6.3.3: Matrix Norms

A norm on a vector space of matrices is called a matrix norm.

**Definition**

For $A \in F^{m \times n}$, the matrix norm induced by the $\ell^p$ vector norm $\| \cdot \|_p$, is:

$$\|A\|_p \triangleq \max_{\|v\|_p = 1} \|Av\|_p.$$  

In special cases, there are exact formulae:

$$\|A\|_\infty = \max_i \sum_j |a_{ij}|$$

$$\|A\|_1 = \max_j \sum_i |a_{ij}|$$

$$\|A\|_2 = \sqrt{\rho(A^HA)},$$

where the $\rho(B)$ is the maximum absolute value of all eigenvalues.

*Examples in live session.*
Sneak Peek: Eigenvalue Decomposition

**Definition**

Let $W$ be a vector space over $F$ and let $T: W \to W$ be a linear operator. A scalar $\lambda \in F$ is an **eigenvalue** of $T$ if there exists a non-zero vector $v \in W$ with $Tv = \lambda v$. Any vector $v$ such that $Tv = \lambda v$ is called an **eigenvector** of $T$ associated with the eigenvalue value $\lambda$.

**Definition**

A matrix $A \in F^{n \times n}$ is **diagonalizable** if there is an invertible matrix $V \in F^{n \times n}$ (whose columns are eigenvectors) such that $V^{-1}AV = \Lambda$ is diagonal.

$$
A \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}
$$

Example of $A^m$ in live session.
6.3.2: Neumann Expansion

**Theorem**

Let $\| \cdot \|$ be a submultiplicative operator norm on the space $L(V, V)$ and let $T \in L(V, V)$ be a linear operator with $\| T \| < 1$. Then, $(I - T)^{-1}$ exists and

$$(I - T)^{-1} = \sum_{i=0}^{\infty} T^i.$$ 

**Proof.**

- Observe that $\sum_{i=0}^{\infty} \| T^i \| \leq \sum_{i=0}^{\infty} \| T \|^i = \frac{1}{1 - \| T \|}$ because $\| T \| < 1$
- Recall an infinite sum $\sum_{i=0}^{\infty} T^i$ converges if $\sum_{i=0}^{\infty} \| T^i \|$ converges
- Next, observe that $(I - T) \left( I + T + T^2 + \cdots + T^{n-1} \right) = I - T^n$
- $T^n \to 0$ because $\| T^n \| \leq \| T \|^n \to 0$ since $\| T \| < 1$
- Thus, $\sum_{i=0}^{\infty} T^i$ is a right inverse for $I - T$
- Same argument works on the left, so we're done.
Next Steps

- To continue studying after this video –
  - Try the required reading: Course Notes EF 3.5, 6.3
  - Or the recommended reading: MMA 4.1-4.2
  - Also, look at the problems in Assignment 5