ECE 586: Vector Space Methods
Lecture 14 Flip Video: Matrix and Operator Norms

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Definition

Let $L(V, W)$ denote the vector space of all linear transforms from V into W, where V and W are vector spaces over a field F.

An operator norm is a norm on a vector space of linear transforms.

Definition (Induced Operator Norm)

Let V and W be two normed vector spaces and let $T\colon V \to W$ be a linear transformation. The induced operator norm of T is defined to

$$\|T\|_{\mathrm{op}} = \sup_{\underline{v} \in V - \{\underline{0}\}} \frac{\|T\underline{v}\|_W}{\|\underline{v}\|_V} = \sup_{\underline{v} \in V, \|\underline{v}\|_V = 1} \|T\underline{v}\|_W .$$

A common question about the operator norm is, "How do I know the two expressions give the same result?". To see this, we can write

$$\sup_{\underline{v} \in V - \{\underline{0}\}} \frac{\|T\underline{v}\|_W}{\|\underline{v}\|_V} = \sup_{\underline{v} \in V - \{\underline{0}\}} \left\| T \frac{\underline{v}}{\|\underline{v}\|_V} \right\|_W = \sup_{\underline{u} \in V, \|\underline{u}\|_V = 1} \|T\underline{u}\|_W .$$

The induced operator norm has another property:

$$\|T\| = \sup_{\underline{v} \in V - \{\underline{0}\}} \frac{\|T\underline{v}\|}{\|\underline{v}\|} \geq \frac{\|T\underline{u}\|}{\|\underline{u}\|},$$

for non-zero $\underline{u} \in V$. This implies that $\|T\underline{u}\| \leq \|T\|\|\underline{u}\|$ for all non-zero $\underline{u} \in V$. By checking $\underline{u} = \underline{0}$ separately, one can see it holds for all $\underline{u} \in V$.

Definition

For the space $L(V, V)$ of linear operators on V, a norm is called submultiplicative if $\|TU\| \leq \|T\|\|U\|$ for all $T, U \in L(V, V)$.

Using the above inequality, induced operator norms are submultiplicative:

$$\|UT\underline{v}\| \leq \|U\| \, \|T\underline{v}\| \leq \|U\| \, \|T\| \, \|\underline{v}\| \, .$$

Continuity of Linear Transforms

Definition

A linear transform is called bounded if its induced operator norm is finite.

Theorem

A linear transformation $T \colon V \to W$ is bounded if and only if it is continuous.

Proof.

Suppose that T is bounded; that is, there exists M such that $\|T\underline{v}\| \le M\|\underline{v}\|$ for all $\underline{v} \in V$. Let $\underline{v}_1, \underline{v}_2, \ldots$ be a convergent sequence in V, then

$$\left\|T\underline{v}_i - T\underline{v}_j\right\| = \left\|T\left(\underline{v}_i - \underline{v}_j\right)\right\| \le M\left\|\underline{v}_i - \underline{v}_j\right\|.$$

This implies $T\underline{v}_1, T\underline{v}_2, \ldots$ is convergent in W and, thus, T is continuous.

Conversely, assume T is continuous and notice that $T\underline{0} = \underline{0}$. Then, for any $\epsilon > 0$, there is a $\delta > 0$ such that $\|T\underline{v}\| < \epsilon$ for all $\|\underline{v}\| < \delta$. It follows that

$$\|T\underline{u}\| = \left\|T\frac{\delta\underline{u}}{2\|\underline{u}\|}\right\| \frac{2\|\underline{u}\|}{\delta} < \frac{2\epsilon}{\delta}\|\underline{u}\|.$$

Thus, $M = \frac{2\epsilon}{\delta}$ serves as an upper bound on $\|T\|$. $\square$

6.3.3: Matrix Norms

A norm on a vector space of matrices is called a matrix norm.

Definition

For $A \in F^{m \times n}$, the matrix norm induced by the ℓ^p vector norm $\|\cdot\|_p$, is:

$$\|A\|_p \triangleq \max_{\|\underline{v}\|_p = 1} \|A\underline{v}\|_p .$$

In special cases, there are exact formulae:

$$\|A\|_\infty = \max_i \sum_j |a_{ij}|$$

$$\|A\|_1 = \max_j \sum_i |a_{ij}|$$

$$\|A\|_2 = \sqrt{\rho(A^H A)},$$

where the $\rho(B)$ is the maximum absolute value of all eigenvalues.

Examples in live session.

Sneak Peek: Eigenvalue Decomposition

Definition

Let W be a vector space over F and let $T\colon W \to W$ be a linear operator. A scalar $\lambda \in F$ is an eigenvalue of T if there exists a non-zero vector $\underline{v} \in W$ with $T\underline{v} = \lambda\underline{v}$. Any vector $\underline{v}$ such that $T\underline{v} = \lambda\underline{v}$ is called an eigenvector of T associated with the eigenvalue value λ.

Definition

A matrix $A \in F^{n\times n}$ is diagonalizable if there is an invertible matrix $V \in F^{n\times n}$ (whose columns are eigenvectors) such that $V^{-1}AV = \Lambda$ is diagonal.

$$A \underbrace{\begin{bmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{bmatrix}}_{V} = \begin{bmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{\Lambda}$$

Example of A^m in live session.

6.3.2: Neumann Expansion

Theorem

Let $\|\cdot\|$ be a submultiplicative operator norm on the space $L(V,V)$ and let $T \in L(V,V)$ be a linear operator with $\|T\| < 1$. Then, $(I-T)^{-1}$ exists and

$$(I-T)^{-1} = \sum_{i=0}^{\infty} T^i.$$

Proof.

- Observe that $\sum_{i=0}^{\infty} \|T^i\| \leq \sum_{i=0}^{\infty} \|T\|^i = \frac{1}{1-\|T\|}$ because $\|T\| < 1$
- Recall an infinite sum $\sum_{i=0}^{\infty} T^i$ converges if $\sum_{i=0}^{\infty} \|T^i\|$ converges
- Next, observe that $(I-T)\left(I + T + T^2 + \cdots + T^{n-1}\right) = I - T^n$
- $T^n \to 0$ because $\|T^n\| \leq \|T\|^n \to 0$ since $\|T\| < 1$
- Thus, $\sum_{i=0}^{\infty} T^i$ is a right inverse for $I - T$
- Same argument works on the left, so we're done. $\square$

Next Steps

- To continue studying after this video –
  - Try the required reading: Course Notes EF 3.5, 6.3
  - Or the recommended reading: MMA 4.1-4.2
  - Also, look at the problems in Assignment 5