Definition

Let $F$ be the field of real numbers or the field of complex numbers, and assume $V$ is a vector space over $F$. An **inner product** on $V$ is a function which assigns to each ordered pair of vectors $v, w \in V$ a scalar $\langle v, w \rangle \in F$ in such a way that for all $u, v, w \in V$ and any scalar $s \in F$

1. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
2. $\langle sv, w \rangle = s \langle v, w \rangle$
3. $\langle v, w \rangle = \overline{\langle w, v \rangle}$, where the overbar denotes complex conjugation;
4. $\langle v, v \rangle \geq 0$ with equality iff $v = 0$.

Note that these conditions imply that:

$$\langle sv + w, u \rangle = s \langle v, u \rangle + \langle w, u \rangle$$

$$\langle u, sv + w \rangle = \overline{s} \langle u, v \rangle + \langle u, w \rangle$$
3.6: Example Inner Products

**Example (Standard Inner Product on $F^n$)**

Consider the inner product on $F^n$ defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \langle (v_1, \ldots, v_n), (w_1, \ldots, w_n) \rangle \triangleq \sum_{j=1}^{n} v_j \bar{w}_j.$$  

For column vectors, it follows that $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{w}^H \mathbf{v}$

**Example (Standard Inner Product on a Function Space)**

Let $V$ be the vector space of all continuous complex-valued functions on the unit interval $[0, 1]$. Then, the following defines an inner product

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} \, dt$$

**Example (Inner Product on Space of Random Variables)**

Let $W$ be a set of real-valued random variables with finite 2nd moments. Then, $V = \text{span}(W)$ is a vector space over $\mathbb{R}$ with inner product

$$\langle X, Y \rangle = \mathbb{E}[XY]$$
Theorem

Let $V$ be a finite-dimensional space with ordered basis $\mathcal{B} = w_1, \ldots, w_n$. Then, any inner product on $V$ is determined by the values $g_{ij} = \langle w_j, w_i \rangle$.

Proof.

If $u = \sum_j s_j w_j$ and $v = \sum_i t_i w_i$, then

$$\langle u, v \rangle = \langle \sum_j s_j w_j, v \rangle = \sum_j s_j \langle w_j, v \rangle = \sum_j s_j \sum_i t_i \langle w_j, w_i \rangle = \sum_j \sum_i s_j t_i \langle w_j, w_i \rangle = \sum_j \sum_i t_i g_{ij} s_j = [v]_\mathcal{B}^H G [u]_\mathcal{B}$$

where $[u]_\mathcal{B} = (s_1, \ldots, s_n)$ and $[v]_\mathcal{B} = (t_1, \ldots, t_n)$ are the coordinate matrices of $u, v$ in the ordered basis $\mathcal{B}$. The matrix $G$ is called the weight matrix of the inner product in the ordered basis $\mathcal{B}$. \qed
3.6: Properties of the Inner Product (2)

- Since $g_{ij} = \langle w_j, w_i \rangle = \overline{\langle w_i, w_j \rangle} = g_{ji}$, we see that
  - The weight matrix $G$ of an inner product is Hermitian: $G = G^H$

- Using $\langle v, v \rangle \geq 0$, we see that $\langle v, v \rangle = w^H G w > 0$ for all $w \neq 0$
  - A Hermitian matrix satisfying this is called positive definite

- If $G$ is an $n \times n$ matrix that is Hermitian and positive definite, then:
  - The following expression is a well-defined inner product on $V$:
    \[
    \langle u, v \rangle_G = [v]_B^H G [u]_B.
    \]

**Definition (Orthogonal)**

Let $v$ and $w$ be vectors in inner-product space $V$. Then, $v$ is orthogonal to $w$ (denoted $v \perp w$) iff $\langle v, w \rangle = 0$. Since this implies $\langle w, v \rangle = 0$, $w$ is also orthogonal to $v$, we simply say that $v$ and $w$ are orthogonal.
3.6.1: Induced Norm

**Definition (Induced Norm)**

Let $V$ be an inner-product space with inner product $\langle \cdot , \cdot \rangle$. This inner product naturally defines the **induced norm**

$$\|v\| = \langle v, v \rangle^{\frac{1}{2}}.$$

**Definition (Projection)**

Let $w, v$ be vectors in an inner-product space $V$ with inner product $\langle \cdot , \cdot \rangle$. The **projection** of $w$ onto $v$ is defined to be

$$u = \frac{\langle w, v \rangle}{\|v\|^2} v.$$
3.6.1: Projection Lemma

**Lemma**

Let $u$ be the projection of $w$ onto $v$. Then, $\langle w - u, u \rangle = 0$ and

$$||w - u||^2 = ||w||^2 - ||u||^2 = ||w||^2 - \frac{||w, v||^2}{||v||^2}.$$  

**Proof.**

First, we observe that

$$\langle w - u, v \rangle = \langle w, v \rangle - \langle u, v \rangle = \langle w, v \rangle - \frac{||w, v||}{||v||^2} \langle v, v \rangle = 0.$$  

Since $u = sv$ for some scalar $s$, it follows that $\langle w - u, u \rangle = 0$. Using $\langle w - u, u \rangle = 0$, we can write

$$||w||^2 = ||w - u + u||^2 = \langle (w - u) + u, (w - u) + u \rangle$$

$$= ||w - u||^2 + 2\text{Re}\langle w - u, u \rangle + ||u||^2 = ||w - u||^2 + ||u||^2.$$  

The proof is completed by noting that $||u||^2 = \frac{||w, v||^2}{||v||^2}$. □
3.6.1: Properties of the Induced Norm

Theorem

If $V$ is an inner-product space over $F$ and $\|v\| \triangleq \sqrt{\langle v, v \rangle}$, then for any $v, w \in V$ and any $s \in F$, it follows that

1. $\|sv\| = |s| \|v\|$
2. $\|v\| > 0$ for $v \neq 0$
3. $|\langle v, w \rangle| \leq \|v\| \|w\|$ with equality iff $v = 0$, $w = 0$, or $v = sw$
4. $\|v + w\| \leq \|v\| + \|w\|$ with equality iff $v = 0$, $w = 0$, or $v = sw$.

Sketch of Proof.

The first two follow immediately from definitions. The third inequality, $|\langle v, w \rangle| \leq \|v\| \|w\|$, is called the Cauchy-Schwarz inequality. The fourth inequality is the triangle inequality for the induced norm and can be shown using the Cauchy-Schwarz inequality.

Proof of Cauchy-Schwarz in live session.
3.7: Sets of Orthogonal Vectors

**Definition**
Let $V$ be an inner-product space and $U, W$ be subspaces. Then, the subspace $U$ is an orthogonal to the subspace $W$ (denoted $U \perp W$) if:

$$u \perp w \text{ for all } u \in U \text{ and } w \in W.$$  

**Definition**
A subset $W \subset V$ of vectors is an orthogonal set if all distinct pairs in $W$ are orthogonal. A orthogonal set is orthonormal if all vectors normalized.

**Example**
For $\mathbb{R}^n$ with standard inner product, the standard basis is an orthonormal.

**Example**
Let $V$ be the vector space (over $\mathbb{C}$) of continuous functions $f : [0, 1] \rightarrow \mathbb{C}$ with the standard inner product. Let $f_n(x) = \sqrt{2} \cos 2\pi nx$ and $g_n(x) = \sqrt{2} \sin 2\pi nx$. Then, $\{1, f_1, g_1, f_2, g_2, \ldots\}$ is a countably infinite orthonormal set and a Schauder basis for the closure of $V$. 
3.7: Properties of Orthogonal Sets

Lemma

Let $V$ be an inner-product space and $W \subset V$ be an orthogonal set of non-zero vectors. Let $v = s_1w_1 + \cdots + s_nw_n$ be a linear combination of distinct vectors in $W$. Then,

$$s_i = \frac{\langle v, w_i \rangle}{\|w_i\|^2}$$

Proof.

The inner product $\langle v, w_i \rangle$ is given by

$$\langle v, w_i \rangle = \left\langle \sum_j s_jw_j, w_i \right\rangle = \sum_j s_j \langle w_j, w_i \rangle = s_i \langle w_i, w_i \rangle.$$

Dividing both sides by $\|w_i\|^2 = \langle w_i, w_i \rangle > 0$, gives the stated result. \(\square\)

Theorem

An orthogonal set of non-zero vectors is linearly independent.

Proof by contradiction in live session.
Gram-Schmidt Orthogonalization (1)

Gram-Schmidt Process

Let $V$ be an inner-product space and assume $v_1, \ldots, v_n$ are linearly independent vectors in $V$. Then, an orthogonal set of vectors $w_1, \ldots, w_n$ with the same span is produced by Gram-Schmidt process:

1. Let $w_1 = v_1$
2. For $m = 1, \ldots, n - 1$, define

$$w_{m+1} = v_{m+1} - \sum_{i=1}^{m} \frac{\langle v_{m+1}, w_i \rangle}{\|w_i\|^2} w_i.$$

- Vector $w_{m+1} \neq 0$. Otherwise, $v_{m+1}$ is a linear combination of $w_1, \ldots, w_m$ and hence a linear combination of $v_1, \ldots, v_m$
- Vectors $w_{m+1}$ and $w_j$ are orthogonal for $j = 1, \ldots, m$:

$$\langle w_{m+1}, w_j \rangle = \langle v_{m+1}, w_j \rangle - \sum_{i=1}^{m} \frac{\langle v_{m+1}, w_i \rangle}{\|w_i\|^2} \langle w_i, w_j \rangle$$

$$= \langle v_{m+1}, w_j \rangle - \frac{\langle v_{m+1}, w_j \rangle}{\|w_j\|^2} \langle w_j, w_j \rangle = 0$$
Example

Let \( V = \mathbb{R}^3 \) be the standard vector space equipped with the standard inner product and define

\[
\begin{align*}
\mathbf{v}_1 &= (2, 2, 1) \\
\mathbf{v}_2 &= (3, 6, 0) \\
\mathbf{v}_3 &= (6, 3, 9)
\end{align*}
\]

Applying the Gram-Schmidt process to \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) results in:

\[
\begin{align*}
\mathbf{w}_1 &= (2, 2, 1) \\
\mathbf{w}_2 &= (3, 6, 0) - \frac{\langle (3, 6, 0), (2, 2, 1) \rangle}{9} (2, 2, 1) \\
&= (3, 6, 0) - 2(2, 2, 1) = (−1, 2, −2) \\
\mathbf{w}_3 &= \mathbf{v}_3 - \frac{\langle (6, 3, 9), (2, 2, 1) \rangle}{9} (2, 2, 1) - \frac{\langle (6, 3, 9), (−1, 2, −2) \rangle}{9} (−1, 2, −2) \\
&= (6, 3, 9) - 3(2, 2, 1) + 2(−1, 2, −2) = (−2, 1, 2)
\end{align*}
\]

It is easily verified that \( \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} \) is an orthogonal set of vectors.
3.7: Orthogonal Complement

**Definition**

Let $V$ be an inner-product space and $W$ be any set of vectors in $V$. The **orthogonal complement** of $W$ denoted by $W^\perp$ is the set of all vectors in $V$ that are orthogonal to every vector in $W$ or

$$W^\perp = \{ v \in V | \langle v, w \rangle = 0 \ \forall \ w \in W \}. $$

**Example**

For the standard inner product space $V = \mathbb{R}^3$, let subspace $U$ be spanned by

$$u_1 = (2, 2, 1)$$
$$u_2 = (3, 6, 0).$$

Find the orthogonal complement $U^\perp$ of $U$.

Discussion in live session.
3.7.1: Hilbert Spaces

Definition
A complete inner-product space is called a Hilbert space.

Example
Consider the Banach space \( \ell^2 \) of infinite real/complex sequences with norm \( \|v\| = (\sum_{i=1}^{\infty} |v_i|^2)^{1/2} < \infty \). The set \( \ell^2 \) with the standard inner product is a Hilbert space because that norm is induced by the inner product.

Theorem
If Hilbert space \( V \) has a countable dense subset, then there is a linear transform \( T : V \to \ell^2 \) such that \( \langle u, v \rangle_V = \langle Tu, Tv \rangle_{\ell^2} \) for all \( u, v \in V \).

Thus, any separable Hilbert space is equivalent to the Hilbert space \( \ell^2 \).
3.7: Unitary Matrices

**Definition**

A complex matrix $U \in \mathbb{C}^{n \times n}$ is called **unitary** if $U^H U = I$. Similarly, a real matrix $Q \in \mathbb{R}^{n \times n}$ is called **orthogonal** if $Q^T Q = I$.

**Theorem**

Let $V = \mathbb{C}^n$ be the standard inner product space and let $U \in \mathbb{C}^{n \times n}$ define a linear operator on $V$. Then, the following conditions are equivalent:

1. The columns of $U$ form an orthonormal basis (i.e., $U^H U = I$),
2. the rows of $U$ form an orthonormal basis (i.e., $UU^H = I$),
3. $U$ preserves inner products (i.e., $\langle Uv, Uw \rangle = \langle v, w \rangle$ for all $u, v \in V$),
4. $U$ is an isometry (i.e., $\|Uv\| = \|v\|$ for all $v \in V$).

**Proof.**

$(i) \Rightarrow (ii)$: Orthogonal columns implies $U$ invertible and $U^H U = I$ implies $U^H = U^{-1}$. $(i) \Rightarrow (iii)$: $\langle Uv, Uw \rangle = w^H U^H Uv = w^H v = \langle v, w \rangle$ and $w = v$ gives $(iv)$. $(iv) \Rightarrow (i)$: $v^H (U^H U - I)v = \|Uv\|^2 - \|v\|^2 = 0$ and, since $U^H U - I$ is Hermitian, all eigenvalues must be 0 so that $U^H U - I = 0$. \(\square\)
3.8: Linear Functionals and the Riesz Theorem

**Definition**
Let $V$ be a vector space over a field $F$. A linear transformation $f$ from $V$ into the scalar field $F$ is called a linear functional on $V$.

**Example**
Thus, $f : V \rightarrow F$ is a function on $V$ such that

$$f(sv_1 + v_2) = sf(v_1) + f(v_2)$$

for all $v_1, v_2 \in V$ and $s \in F$.

**Theorem (Riesz)**
Let $V$ be a Hilbert space and $f$ be a continuous linear functional on $V$. Then, there exists a unique vector $v \in V$ such that $f(w) = \langle w, v \rangle$ for all $w \in V$. 
To continue studying after this video –

- Try the required reading: Course Notes EF 3.6 - 3.8
- Or the recommended reading: LADR 6AB
- Also, look at the problems in Assignment 6