ECE 586: Vector Space Methods
Lecture 15 Flip Video: Inner Product Spaces

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3.6: Inner-Product Spaces

Definition

Let F be the field of real numbers or the field of complex numbers, and assume V is a vector space over F. An inner product on V is a function which assigns to each ordered pair of vectors $\underline{v}, \underline{w} \in V$ a scalar $\langle \underline{v}, \underline{w} \rangle \in F$ in such a way that for all $\underline{u}, \underline{v}, \underline{w} \in V$ and any scalar $s \in F$

1. $\langle \underline{u} + \underline{v}, \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle$
2. $\langle s\underline{v}, \underline{w} \rangle = s\langle \underline{v}, \underline{w} \rangle$
3. $\langle \underline{v}, \underline{w} \rangle = \overline{\langle \underline{w}, \underline{v} \rangle}$, where the overbar denotes complex conjugation;
4. $\langle \underline{v}, \underline{v} \rangle \geq 0$ with equality iff $\underline{v} = \underline{0}$.

Note that these conditions imply that:

$$\langle s\underline{v} + \underline{w}, \underline{u} \rangle = s\langle \underline{v}, \underline{u} \rangle + \langle \underline{w}, \underline{u} \rangle$$
$$\langle \underline{u}, s\underline{v} + \underline{w} \rangle = \bar{s}\langle \underline{u}, \underline{v} \rangle + \langle \underline{u}, \underline{w} \rangle$$

3.6: Example Inner Products

Example (Standard Inner Product on F^n)

Consider the inner product on F^n defined by

$$\langle \underline{v}, \underline{w} \rangle = \langle (v_1, \ldots, v_n), (w_1, \ldots, w_n) \rangle \triangleq \sum_{j=1}^{n} v_j \overline{w}_j .$$

For column vectors, it follows that $\langle \underline{v}, \underline{w} \rangle = \underline{w}^H \underline{v}$

Example (Standard Inner Product on a Function Space)

Let V be the vector space of all continuous complex-valued functions on the unit interval $[0, 1]$. Then, the following defines an inner product

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)}\, dt$$

Example (Inner Product on Space of Random Variables)

Let W be a set of real-valued random variables with finite 2nd moments. Then, $V = \mathrm{span}(W)$ is a vector space over $\mathbb{R}$ with inner product

$$\langle X, Y \rangle = \mathrm{E}\,[XY]$$

3.6: Properties of the Inner Product (1)

Theorem

Let V be a finite-dimensional space with ordered basis $\mathcal{B} = \underline{w}_1, \ldots, \underline{w}_n$. Then, any inner product on V is determined by the values

$$g_{ij} = \langle \underline{w}_j \, , \underline{w}_i \rangle.$$

Proof.

If $\underline{u} = \sum_j s_j \underline{w}_j$ and $\underline{v} = \sum_i t_i \underline{w}_i$, then

$$\begin{aligned} \langle \underline{u} \, , \underline{v} \rangle &= \langle \textstyle\sum_j s_j \underline{w}_j \, , \underline{v} \rangle = \textstyle\sum_j s_j \langle \underline{w}_j \, , \underline{v} \rangle \\ &= \textstyle\sum_j s_j \langle \underline{w}_j \, , \textstyle\sum_i t_i \underline{w}_i \rangle = \textstyle\sum_j \textstyle\sum_i s_j \bar{t}_i \langle \underline{w}_j \, , \underline{w}_i \rangle \\ &= \textstyle\sum_j \textstyle\sum_i \bar{t}_i g_{ij} s_j = [\underline{v}]_{\mathcal{B}}^H \, G \, [\underline{u}]_{\mathcal{B}} \end{aligned}$$

where $[\underline{u}]_{\mathcal{B}} = (s_1, \ldots, s_n)$ and $[\underline{v}]_{\mathcal{B}} = (t_1, \ldots, t_n)$ are the coordinate matrices of $\underline{u}$, $\underline{v}$ in the ordered basis $\mathcal{B}$. The matrix G is called the weight matrix of the inner product in the ordered basis $\mathcal{B}$. □

3.6: Properties of the Inner Product (2)

- Since $g_{ij} = \left\langle \underline{w}_j , \underline{w}_i \right\rangle = \overline{\left\langle \underline{w}_i , \underline{w}_j \right\rangle} = \overline{g_{ji}}$, we see that
  - The weight matrix G of an inner product is Hermitian: $G = G^H$
- Using $\langle \underline{v} , \underline{v} \rangle \geq 0$, we see that $\langle \underline{v} , \underline{v} \rangle = \underline{w}^H G \underline{w} > 0$ for all $\underline{w} \neq \underline{0}$
  - A Hermitian matrix satisfying this is called positive definite
- If G is an $n \times n$ matrix that is Hermitian and positive definite, then:
  - The following expression is a well-defined inner product on V:

$$\langle \underline{u} , \underline{v} \rangle_G = [\underline{v}]_{\mathcal{B}}^H \, G \, [\underline{u}]_{\mathcal{B}} \, .$$

Definition (Orthogonal)

Let $\underline{v}$ and $\underline{w}$ be vectors in inner-product space V. Then, $\underline{v}$ is orthogonal to $\underline{w}$ (denoted $\underline{v} \perp \underline{w}$) iff $\langle \underline{v} , \underline{w} \rangle = 0$. Since this implies $\langle \underline{w} , \underline{v} \rangle = 0$, $\underline{w}$ is also orthogonal to $\underline{v}$, we simply say that $\underline{v}$ and $\underline{w}$ are orthogonal.

3.6.1: Induced Norm

Definition (Induced Norm)

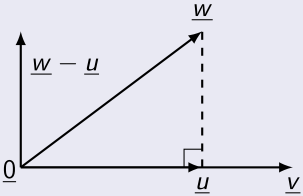
Let V be an inner-product space with inner product $\langle \cdot, \cdot \rangle$. This inner product naturally defines the induced norm

$$\|\underline{v}\| = \langle \underline{v}, \underline{v} \rangle^{\frac{1}{2}}.$$

Definition (Projection)

Let $\underline{w}, \underline{v}$ be vectors in an inner-product space V with inner product $\langle \cdot, \cdot \rangle$. The projection of $\underline{w}$ onto $\underline{v}$ is defined to be

$$\underline{u} = \frac{\langle \underline{w}, \underline{v} \rangle}{\|\underline{v}\|^2} \underline{v}$$



3.6.1: Projection Lemma

Lemma

Let $\underline{u}$ *be the projection of* $\underline{w}$ *onto* $\underline{v}$. *Then,* $\langle \underline{w} - \underline{u}, \underline{u} \rangle = 0$ *and*

$$\|\underline{w} - \underline{u}\|^2 = \|\underline{w}\|^2 - \|\underline{u}\|^2 = \|\underline{w}\|^2 - \frac{|\langle \underline{w}, \underline{v} \rangle|^2}{\|\underline{v}\|^2}.$$

Proof.

First, we observe that

$$\langle \underline{w} - \underline{u}, \underline{v} \rangle = \langle \underline{w}, \underline{v} \rangle - \langle \underline{u}, \underline{v} \rangle = \langle \underline{w}, \underline{v} \rangle - \frac{\langle \underline{w}, \underline{v} \rangle}{\|\underline{v}\|^2} \langle \underline{v}, \underline{v} \rangle = 0.$$

Since $\underline{u} = s\underline{v}$ for some scalar s, it follows that $\langle \underline{w} - \underline{u}, \underline{u} \rangle = 0$. Using $\langle \underline{w} - \underline{u}, \underline{u} \rangle = 0$, we can write

$$\begin{aligned} \|\underline{w}\|^2 &= \|(\underline{w} - \underline{u}) + \underline{u}\|^2 = \langle (\underline{w} - \underline{u}) + \underline{u}, (\underline{w} - \underline{u}) + \underline{u} \rangle \\ &= \|\underline{w} - \underline{u}\|^2 + 2\mathrm{Re}\langle \underline{w} - \underline{u}, \underline{u} \rangle + \|\underline{u}\|^2 = \|\underline{w} - \underline{u}\|^2 + \|\underline{u}\|^2. \end{aligned}$$

The proof is completed by noting that $\|\underline{u}\|^2 = |\langle \underline{w}, \underline{v} \rangle|^2 / \|\underline{v}\|^2$. □

3.6.1: Properties of the Induced Norm

Theorem

If V *is an inner-product space over* F *and* $\|\underline{v}\| \triangleq \sqrt{\langle \underline{v}, \underline{v} \rangle}$, *then for any* $\underline{v}, \underline{w} \in V$ *and any* $s \in F$, *it follows that*

1. $\|s\underline{v}\| = |s| \, \|\underline{v}\|$
2. $\|\underline{v}\| > 0$ *for* $\underline{v} \neq \underline{0}$
3. $|\langle \underline{v}, \underline{w} \rangle| \leq \|\underline{v}\| \, \|\underline{w}\|$ *with equality iff* $\underline{v} = \underline{0}$, $\underline{w} = \underline{0}$, *or* $\underline{v} = s\underline{w}$
4. $\|\underline{v} + \underline{w}\| \leq \|\underline{v}\| + \|\underline{w}\|$ *with equality iff* $\underline{v} = \underline{0}$, $\underline{w} = \underline{0}$, *or* $\underline{v} = s\underline{w}$.

Sketch of Proof.

The first two follow immediately from definitions. The third inequality, $|\langle \underline{v}, \underline{w} \rangle| \leq \|\underline{v}\| \, \|\underline{w}\|$, is called the Cauchy-Schwarz inequality. The fourth inequality is the triangle inequality for the induced norm and can be shown using the Cauchy-Schwarz inequality. □

Proof of Cauchy-Schwarz in live session.

3.7: Sets of Orthogonal Vectors

Definition

Let V be an inner-product space and U, W be subspaces. Then, the subspace U is an orthogonal to the subspace W (denoted $U \perp W$) if:

$$\underline{u} \perp \underline{w} \text{ for all } \underline{u} \in U \text{ and } \underline{w} \in W.$$

Definition

A subset $W \subset V$ of vectors is an orthogonal set if all distinct pairs in W are orthogonal. A orthogonal set is orthonormal if all vectors normalized.

Example

For $\mathbb{R}^n$ with standard inner product, the standard basis is an orthonormal.

Example

Let V be the vector space (over $\mathbb{C}$) of continuous functions $f : [0, 1] \to \mathbb{C}$ with the standard inner product. Let $f_n(x) = \sqrt{2} \cos 2\pi nx$ and $g_n(x) = \sqrt{2} \sin 2\pi nx$. Then, $\{1, f_1, g_1, f_2, g_2, \ldots\}$ is a countably infinite orthonormal set and a Schauder basis for the closure of V.

3.7: Properties of Orthogonal Sets

Lemma

Let V be an inner-product space and $W \subset V$ be an orthogonal set of non-zero vectors. Let $\underline{v} = s_1 \underline{w}_1 + \cdots + s_n \underline{w}_n$ be a linear combination of distinct vectors in W. Then,

$$s_i = \frac{\langle \underline{v}, \underline{w}_i \rangle}{\|\underline{w}_i\|^2}$$

Proof.

The inner product $\langle \underline{v}, \underline{w}_i \rangle$ is given by

$$\langle \underline{v}, \underline{w}_i \rangle = \left\langle \sum_j s_j \underline{w}_j, \underline{w}_i \right\rangle = \sum_j s_j \langle \underline{w}_j, \underline{w}_i \rangle = s_i \langle \underline{w}_i, \underline{w}_i \rangle.$$

Dividing both sides by $\|\underline{w}_i\|^2 = \langle \underline{w}_i, \underline{w}_i \rangle > 0$, gives the stated result. ☐

Theorem

An orthogonal set of non-zero vectors is linearly independent.

Proof by contradiction in live session.

3.7: Gram-Schmidt Orthogonalization (1)

Gram-Schmidt Process

Let V be an inner-product space and assume $\underline{v}_1, \ldots, \underline{v}_n$ are linearly independent vectors in V. Then, an orthogonal set of vectors $\underline{w}_1, \ldots, \underline{w}_n$ with the same span is produced by Gram-Schmidt process:

1. Let $\underline{w}_1 = \underline{v}_1$
2. For $m = 1, \ldots, n-1$, define

$$\underline{w}_{m+1} = \underline{v}_{m+1} - \sum_{i=1}^{m} \frac{\left\langle \underline{v}_{m+1}, \underline{w}_i \right\rangle}{\left\| \underline{w}_i \right\|^2} \underline{w}_i .$$

- Vector $\underline{w}_{m+1} \neq 0$. Otherwise, $\underline{v}_{m+1}$ is a linear combination of $\underline{w}_1, \ldots, \underline{w}_m$ and hence a linear combination of $\underline{v}_1, \ldots, \underline{v}_m$
- Vectors $\underline{w}_{m+1}$ and $\underline{w}_j$ are orthogonal for $j = 1, \ldots, m$:

$$\begin{aligned} \left\langle \underline{w}_{m+1}, \underline{w}_j \right\rangle &= \left\langle \underline{v}_{m+1}, \underline{w}_j \right\rangle - \sum_{i=1}^{m} \frac{\left\langle \underline{v}_{m+1}, \underline{w}_i \right\rangle}{\left\| \underline{w}_i \right\|^2} \left\langle \underline{w}_i, \underline{w}_j \right\rangle \\ &= \left\langle \underline{v}_{m+1}, \underline{w}_j \right\rangle - \frac{\left\langle \underline{v}_{m+1}, \underline{w}_j \right\rangle}{\left\| \underline{w}_j \right\|^2} \left\langle \underline{w}_j, \underline{w}_j \right\rangle = 0 \end{aligned}$$

3.7: Gram-Schmidt Orthogonalization (2)

Example

Let $V = \mathbb{R}^3$ be the standard vector space equipped with the standard inner product and define

$$\begin{aligned} \underline{v}_1 &= (2,2,1) \\ \underline{v}_2 &= (3,6,0) \\ \underline{v}_3 &= (6,3,9) \end{aligned}$$

Applying the Gram-Schmidt process to $\underline{v}_1, \underline{v}_2, \underline{v}_3$ results in:

$$\begin{aligned} \underline{w}_1 &= (2,2,1) \\ \underline{w}_2 &= (3,6,0) - \frac{\langle (3,6,0), (2,2,1) \rangle}{9}(2,2,1) \\ &= (3,6,0) - 2(2,2,1) = (-1,2,-2) \\ \underline{w}_3 &= \underline{v}_3 - \frac{\langle (6,3,9), (2,2,1) \rangle}{9}(2,2,1) - \frac{\langle (6,3,9), (-1,2,-2) \rangle}{9}(-1,2,-2) \\ &= (6,3,9) - 3(2,2,1) + 2(-1,2,-2) = (-2,1,2) \end{aligned}$$

It is easily verified that $\{\underline{w}_1, \underline{w}_2, \underline{w}_3\}$ is an orthogonal set of vectors.

3.7: Orthogonal Complement

Definition

Let V be an inner-product space and W be any set of vectors in V. The orthogonal complement of W denoted by $W^{\perp}$ is the set of all vectors in V that are orthogonal to every vector in W or

$$W^{\perp} = \left\{ \underline{v} \in V \middle| \langle \underline{v}, \underline{w} \rangle = 0 \; \forall \; \underline{w} \in W \right\}.$$

Example

For the standard inner product space $V = \mathbb{R}^3$, let subspace U be spanned by

$$\underline{u}_1 = (2, 2, 1)$$
$$\underline{u}_2 = (3, 6, 0).$$

Find the orthogonal complement $U^{\perp}$ of U.

Discussion in live session.

3.7.1: Hilbert Spaces

Definition

A complete inner-product space is called a Hilbert space.

Example

Consider the Banach space ℓ^2 of infinite real/complex sequences with norm $\|\underline{v}\| = \left(\sum_{i=1}^{\infty} |v_i|^2\right)^{1/2} < \infty$. The set ℓ^2 with the standard inner product is a Hilbert space because that norm is induced by the inner product.

Theorem

If Hilbert space V has a countable dense subset, then there is a linear transform $T : V \to \ell^2$ such that $\langle \underline{u}, \underline{v} \rangle_V = \langle T\underline{u}, T\underline{v} \rangle_{\ell^2}$ for all $\underline{u}, \underline{v} \in V$.

Thus, any separable Hilbert space is equivalent to the Hilbert space ℓ^2.

3.7: Unitary Matrices

Definition

A complex matrix $U \in \mathbb{C}^{n \times n}$ is called unitary if $U^H U = I$. Similarly, a real matrix $Q \in \mathbb{R}^{n \times n}$ is called orthogonal if $Q^T Q = I$.

Theorem

Let $V = \mathbb{C}^n$ be the standard inner product space and let $U \in \mathbb{C}^{n \times n}$ define a linear operator on V. Then, the following conditions are equivalent:

(i) *The columns of U form an orthonormal basis (i.e., $U^H U = I$),*

(ii) *the rows of U form an orthonormal basis (i.e., $UU^H = I$),*

(iii) *U preserves inner products (i.e., $\langle U\underline{v}, U\underline{w} \rangle = \langle \underline{v}, \underline{w} \rangle$ for all $\underline{u}, \underline{v} \in V$),*

(iv) *U is an isometry (i.e., $\|U\underline{v}\| = \|\underline{v}\|$ for all $\underline{v} \in V$).*

Proof.

(i)⇒(ii): Orthogonal columns implies U invertible and $U^H U = I$ implies $U^H = U^{-1}$. *(i)⇒(iii)*: $\langle U\underline{v}, U\underline{w} \rangle = \underline{w}^H U^H U \underline{v} = \underline{w}^H \underline{v} = \langle \underline{v}, \underline{w} \rangle$ and $\underline{w} = \underline{v}$ gives *(iv)*. *(iv)⇒(i)*: $\underline{v}^H (U^H U - I)\underline{v} = \|U\underline{v}\|^2 - \|\underline{v}\|^2 = 0$ and, since $U^H U - I$ is Hermitian, all eigenvalues must be 0 so that $U^H U - I = 0$. □

3.8: Linear Functionals and the Riesz Theorem

Definition

Let V be a vector space over a field F. A linear transformation f from V into the scalar field F is called a linear functional on V.

Example

Thus, $f : V \to F$ is a function on V such that

$$f\left(s\underline{v}_1 + \underline{v}_2\right) = sf\left(\underline{v}_1\right) + f\left(\underline{v}_2\right)$$

for all $\underline{v}_1, \underline{v}_2 \in V$ and $s \in F$.

Theorem (Riesz)

Let V be a Hilbert space and f be a continuous linear functional on V. Then, there exists a unique vector $\underline{v} \in V$ such that $f\left(\underline{w}\right) = \langle \underline{w}, \underline{v} \rangle$ for all $\underline{w} \in V$.

Next Steps

- To continue studying after this video –
  - Try the required reading: Course Notes EF 3.6 - 3.8
  - Or the recommended reading: LADR 6AB
  - Also, look at the problems in Assignment 6