# ECE 586: Vector Space Methods
# Lecture 19: Four Fundamental Subspaces

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# 6.5: The Four Fundamental Subspaces: Basic Idea

Consider a linear transform mapping $\mathbb{R}^n \to \mathbb{R}^m$ represented by $A \in \mathbb{R}^{m\times n}$

- This $\underline{x} \mapsto A\underline{x}$ mapping defines four fundamental subspaces:
  - The column space or range of $A$: $\mathcal{R}(A)$
  - The row space or range of $A^T$: $\mathcal{R}(A^T)$
  - The null space: $\mathcal{N}(A)$
  - The null space of $A^T$: $\mathcal{N}(A^T)$
- Notice that $\underline{x} \in \mathbb{R}^n$ is in the null space of $A$ *if and only if*

$$A\underline{x} = \begin{bmatrix} --\text{row } 1-- \\ --\text{row } 2-- \\ \vdots \\ --\text{row m}-- \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

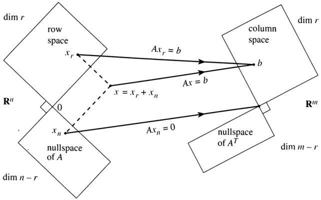


**Figure 1.** The action of $A$: Row space to column space, nullspace to zero.

i.e., all rows of $A$ orthogonal to $\underline{x}$ or $\underline{x} \in \mathcal{R}(A^T)^\perp$

- null space $\mathcal{N}(A)$ is the orthogonal complement $\mathcal{R}(A^T)^\perp$ of row space

- So, null space of $A$ is the orthogonal complement of column space of $A^T$
- Applying the same result to $A^T$ (which maps $\mathbb{R}^m \to \mathbb{R}^n$) shows

  null space of $A^T$ is the orthogonal complement of column space of $A$
- This means that $\mathcal{N}(A) = \mathcal{R}(A^T)^\perp$ and $\mathcal{N}(A^T) = \mathcal{R}(A)^\perp$

**Definition (Adjoint)**

Let $V$ and $W$ be Hilbert spaces, and assume $T\colon V \to W$ is a continuous linear transformation (i.e., $\|T\|_{\mathrm{op}} < \infty$). Then, the adjoint $T^*$ is the unique linear transformation mapping $W$ to $V$ such that

$$\langle T\underline{v}, \underline{w}\rangle_W = \langle \underline{v}, T^*\underline{w}\rangle_V \qquad \text{for all } \underline{v} \in V, \underline{w} \in W$$

The following 3 conditions are equivalent:

- For all $\underline{w} \in W$, we have $\langle \underline{v}, T^*\underline{w}\rangle_V = 0$
- $\underline{v} \in \mathcal{R}(T^*)^\perp$
- $\underline{v} \in \mathcal{N}(T)$ because $\langle \underline{v}, T^*\underline{w}\rangle_V = \langle T\underline{v}, \underline{w}\rangle_W = 0$ for all $\underline{w} \in W$

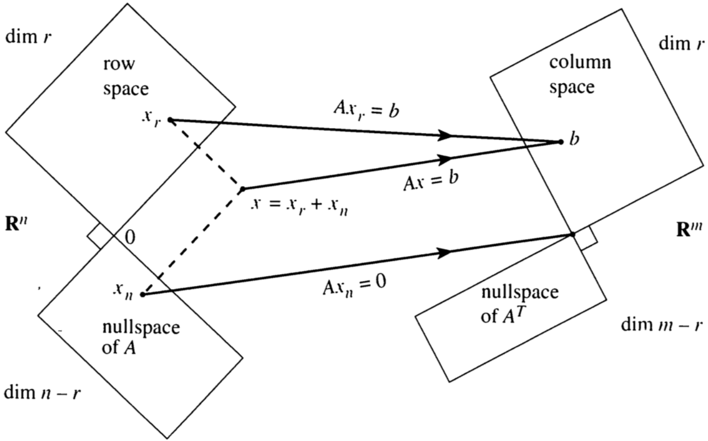


**Figure 1.** The action of $A$: Row space to column space, nullspace to zero.

$r \triangleq \dim(\mathcal{R}(A))$ implies $\dim(\mathcal{N}(A)) = n - r$ and $\dim(\mathcal{N}(A^T)) = m - r$

Figure from "The Fundamental Theorem of Linear Algebra" by Gilbert Strang, The American Mathematical Monthly, Nov. 1993

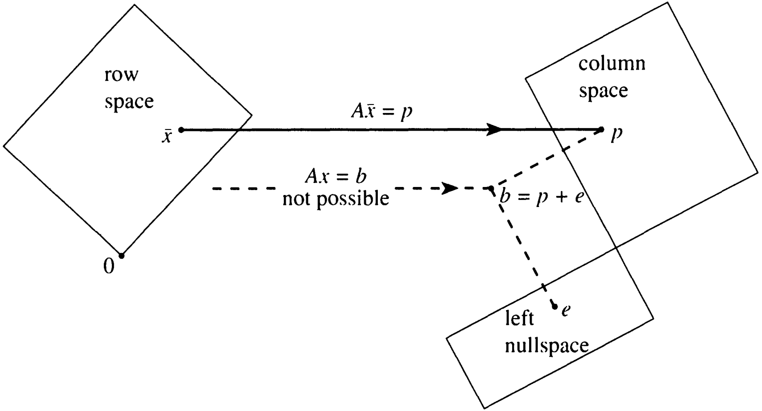


**Figure 2.** Least squares: $\bar{x}$ minimizes $\|b - Ax\|^2$ by solving $A^T A\bar{x} = A^T b$.

Observe $A^T A \colon \mathbb{R}^n \to \mathbb{R}^n$ is invertible if non-singular (i.e., if $n = r$)

Figure from "The Fundamental Theorem of Linear Algebra" by Gilbert Strang, The American Mathematical Monthly, Nov. 1993

# Interlude: Alternative for Linear Systems

A "Theorem of the Alternative" asserts that exactly one of two logical statements is true. The following is a famous example.

**Theorem**

*For a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $\underline{b} \in \mathbb{R}^m$, exactly one of the following statements is true:*

- *There exists an $\underline{x} \in \mathbb{R}^n$ such that $A\underline{x} = \underline{b}$*
- *There exists a $\underline{y} \in \mathbb{R}^m$ such that $A^T\underline{y} = \underline{0}$ and $\underline{y}^T\underline{b} \neq 0$.*

**Proof.**

- In the first case, $\underline{b} \in \mathcal{R}(A)$ and the four fundamental subspaces imply that $\underline{y}^T\underline{b} = 0$ (i.e., $\underline{y} \perp \underline{b}$) if $A^T\underline{y} = \underline{0}$ (i.e., $\underline{y} \in \mathcal{N}(A^T)$).
- In the second case, $\underline{y} \in \mathcal{N}(A^T)$ implies $\underline{y} \perp \mathcal{R}(A)$. So, $\underline{y}^T\underline{b} \neq 0$ implies $\underline{b} \notin \mathcal{R}(A)$. □

# Next Steps

- To continue studying after this video –
  - Try the required reading from website:

    The Fundamental Theorem of Linear Algebra by Gilbert Strang
  - Or the recommended reading: Course Notes EF 6.5 - 6.6.1
  - Also, look at the problems in Assignments 7 and 8