ECE 586: Vector Space Methods
Lecture 2 Flip Video: Predicate Logic

Henry D. Pfister
Duke University

1.2.2: Propositions and Predicates

- The logic we have discussed so far is called propositional logic
- Propositional logic has some limitations
  - Ex. If "Socrates is a person" and "Every person is mortal"
  - Then, we also know that "Socrates is mortal" but, in propositional logic, there is no way to formally deduce this by combining the statements
- To overcome this, we use predicate logic
  - Let U ("the universe") be a collection of elements and $P(x)$ be a statement that can be applied to any $x \in U$
  - Ex. $P(x) =$ "x has 4 tires" for the collection U of all vehicles
  - Statement $P(x)$ is called a predicate and x is called a free variable

1.2.2: Quantifiers

- Quantifiers allow statements about collections of elements
  - Universal quantifier: $\forall x \in U, P(x)$ = "All vehicles have 4 tires"
  - Existential quantifier: $\exists x \in U, P(x)$ = "There is a vehicle with 4 tires"
  - Universal instantiation: if U not empty, then $\forall x \in U, P(x) \Rightarrow \exists x \in U, P(x)$
- Negation
  - Notice exactly one of these is true: $\forall x \in U, P(x)$ or $\exists x \in U, \neg P(x)$
  - Thus, they are logical negations of each other and we observe that

$$\neg(\forall x \in U, P(x)) \Leftrightarrow (\exists x \in U, \neg P(x))$$
$$\neg(\exists x \in U, P(x)) \Leftrightarrow (\forall x \in U, \neg P(x))$$

- Examples
  - "Not all vehicles have 4 tires" $\Leftrightarrow$ "There is a vehicle that does not have 4 tires"
  - "There does not exist a vehicle with 4 tires" $\Leftrightarrow$ "All vehicles do not have 4 tires"

1.2.2: Multiple Quantifiers

- Now, consider the predicate $P(x, y)$ with two free variables, x, y
  - Ex. Let I be a collection of images and C be a collection of colors. Define predicate $P(x, y) =$ "x contains y" for $x \in I$ and $y \in C$
  - "$\forall x \in I, \forall y \in C, P(x, y)$" = "All images in I contain all colors in C"
  - "$\forall x \in I, \exists y \in C, P(x, y)$" = "All images in I contain a color in C"
  - In "$\exists y \in U, P(x, y)$", x is a free variable and y is a bound variable

- Negation
  - Apply one at a time: $\forall x \in I, \forall y \in C, P(x, y) \Leftrightarrow \forall x \in I, (\forall y \in C, P(x, y))$
  - $\neg(\forall x \in I, \forall y \in C, P(x, y)) \Leftrightarrow \exists x \in I, \exists y \in C, \neg P(x, y)$
  - $\neg(\exists x \in I, \forall y \in C, P(x, y)) \Leftrightarrow \forall x \in I, \exists y \in C, \neg P(x, y)$

1.2.2: Relationships Between Multiple Quantifiers

- Implications and Equivalences (assuming I and C are not empty)
  - $\forall x \in I, \forall y \in C, P(x,y) \Leftrightarrow \forall y \in C, \forall x \in I, P(x,y)$
  - $\exists x \in I, \exists y \in C, P(x,y) \Leftrightarrow \exists y \in C, \exists x \in I, P(x,y)$
  - $\forall x \in I, \forall y \in C, P(x,y) \Rightarrow \exists x \in I, \forall y \in C, P(x,y)$ (since I is not empty)
  - $\exists x \in I, \forall y \in C, P(x,y) \Rightarrow \forall y \in C, \exists x \in I, P(x,y)$
  - There are 8 possible choices for the order and type of quantifiers

- These results and their symmetric pairs can be visualized with

$$\begin{array}{ccccccc} \forall x, \forall y, P(x,y) & \Rightarrow & \exists x, \forall y, P(x,y) & \Rightarrow & \forall y, \exists x, P(x,y) & \Rightarrow & \exists y, \exists x, P(x,y) \\ \Updownarrow & & & & & & \Updownarrow \\ \forall y, \forall x, P(x,y) & \Rightarrow & \exists y, \forall x, P(x,y) & \Rightarrow & \forall x, \exists y, P(x,y) & \Rightarrow & \exists x, \exists y, P(x,y) \end{array}$$

Axiomatic Formulations

- "Ex falso quodlibet" is Latin for "from falsehood, anything"
  - Observe that $P \wedge \neg P \rightarrow Q$ is true regardless of Q
  - Thus, logicians are careful to avoid contradictions
  - Fortunately, propositional logic has an axiomatic formulation that is consistent, complete, and decidable
- What do these words mean in logic?
  - Consistent: implications of axioms do not contain a contradiction
  - Complete: all valid implications follow from the axioms
  - Decidable: terminating algorithm determines if any implication is valid or not
- First-Order Predicate Logic
  - Axiomatic formulation is consistent, complete, and semidecidable
  - Semidecidable: algorithm determines the truth of any postulated implication, if it terminates. But, termination is guaranteed only if postulate is true

1.3: Strategies for Proofs

- Background
  - Intuition identifies what might be true and why
  - Rigorous proofs verify and communicate that intuition
  - A proof is a sequence of verifiable steps from assumptions to conclusion
  - Definitions map between words and symbols (e.g., $P(x) =$ "x is even")
- Types of Proofs for $P \to Q$
  - Direct: Assume P true and give steps that lead to Q
  - Contrapositive: Proof of the equivalent statement $\neg Q \to \neg P$
  - Contradiction: Using $\neg(P \to Q) \Leftrightarrow P \land \neg Q$, one supposes that both P and $\neg Q$ are true and then gives steps leading to a contradiction
  - Induction: For predicate $P(n)$, prove $Q =$ "$\forall n \in \mathbb{N}, P(n)$" by establishing the premise $P =$ "$P(1) \land (\forall n \in \mathbb{N}, P(n) \to P(n+1))$"
- Whiteboard Examples
  - Euclid's proof: "if $x^2 = 2$ and x real, then x is not rational" via contradiction
  - For $P(n) =$ "$\sum_{i=1}^{n} i = \frac{n^2+n}{2}$", prove "$\forall n \in \mathbb{N}, P(n)$" via induction

Next Steps

- To continue studying after this video –
  - Try the suggested reading: Course Notes EF 1.2.3-1.3
  - Or the optional reading: PAF 1.1-2.6
  - Also, look at the problems in Assignment 1