

ECE 586: Vector Space Methods
Lecture 2 Flip Video: Predicate Logic

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1.2.2: Propositions and Predicates

- The logic we have discussed so far is called **propositional logic**
- Propositional logic has some limitations
 - Ex. If “Socrates is a person” and “Every person is mortal”
 - Then, we also know that “Socrates is mortal” but, in propositional logic, there is **no way to formally deduce this** by combining the statements
- To overcome this, we use **predicate logic**
 - Let U be a collection of elements and $P(x)$ be a statement that can be applied to any $x \in U$
 - Ex. $P(x) =$ “ x has 4 tires” for the collection U of all vehicles
 - Statement $P(x)$ is called a **predicate** and x is called a **free variable**

1.2.2: Quantifiers

- Quantifiers allow statements about collections of elements
 - Universal quantifier: $\forall x \in U, P(x)$ = “All vehicles have 4 tires”
 - Existential quantifier: $\exists x \in U, P(x)$ = “There is a vehicle with 4 tires”
 - Natural implication: $\forall x \in U, P(x) \Rightarrow \exists x \in U, P(x)$

- Negation

- Notice exactly one of these is true: $\forall x \in U, P(x)$ or $\exists x \in U, \neg P(x)$
- Thus, they are logical negations of each other and we get

$$\neg(\forall x \in U, P(x)) \Leftrightarrow (\exists x \in U, \neg P(x))$$

$$\neg(\exists x \in U, P(x)) \Leftrightarrow (\forall x \in U, \neg P(x))$$

- Examples

- “Not all vehicles have 4 tires” \Leftrightarrow “There is a vehicle that does not have 4 tires”
- “There does not exist a vehicle with 4 tires” \Leftrightarrow “All vehicles do not have 4 tires”

1.2.2: Multiple Quantifiers

- Now, consider the predicate $P(x, y)$ with two free variables, x, y
 - Ex. Let I be a collection of images and C be a collection of colors. Define predicate $P(x, y) = \text{"}x \text{ contains } y\text{"}$ for $x \in I$ and $y \in C$
 - " $\forall x \in I, \forall y \in C, P(x, y)$ " = "All images in I contain all colors in C "
 - " $\forall x \in I, \exists y \in C, P(x, y)$ " = "All images in I contain a color in C "
 - In " $\exists y \in U, P(x, y)$ ", x is a **free variable** and y is a **bound variable**
 - Apply one at a time: $\forall x \in I, \forall y \in C, P(x, y) \Leftrightarrow \forall x \in I, (\forall y \in C, P(x, y))$
- Negation
 - $\neg(\forall x \in I, \forall y \in C, P(x, y)) \Leftrightarrow \exists x \in I, \exists y \in C, \neg P(x, y)$
 - $\neg(\exists x \in I, \forall y \in C, P(x, y)) \Leftrightarrow \forall x \in I, \exists y \in C, \neg P(x, y)$

1.2.2: Relationships Between Multiple Quantifiers

- Implications and Equivalences

- $\forall x \in I, \forall y \in C, P(x, y) \Leftrightarrow \forall y \in C, \forall x \in I, P(x, y)$

- $\exists x \in I, \exists y \in C, P(x, y) \Leftrightarrow \exists y \in C, \exists x \in I, P(x, y)$

- $\forall x \in I, \forall y \in C, P(x, y) \Rightarrow \exists x \in I, \forall y \in C, P(x, y)$

- $\exists x \in I, \forall y \in C, P(x, y) \Rightarrow \forall y \in C, \exists x \in I, P(x, y)$

- There are 8 possible choices for the order and quantifiers

- These results and their symmetric pairs can be visualized with

$$\begin{array}{ccccccc} \forall x, \forall y, P(x, y) & \Rightarrow & \exists x, \forall y, P(x, y) & \Rightarrow & \forall y, \exists x, P(x, y) & \Rightarrow & \exists y, \exists x, P(x, y) \\ & & \updownarrow & & & & \updownarrow \\ \forall y, \forall x, P(x, y) & \Rightarrow & \exists y, \forall x, P(x, y) & \Rightarrow & \forall x, \exists y, P(x, y) & \Rightarrow & \exists x, \exists y, P(x, y) \end{array}$$

Axiomatic Formulations

- “Ex falso quodlibet” is Latin for “from falsehood, anything”
 - Observe that $P \wedge \neg P \rightarrow Q$ is true regardless of Q
 - Thus, logicians are careful to avoid contradictions
 - Fortunately, propositional logic has an axiomatic formulation that is consistent, complete, and decidable
- What do these words mean in logic?
 - **Consistent**: implications of axioms do not contain a contradiction
 - **Complete**: all valid implications follow from the axioms
 - **Decidable**: algorithm can determine if any implication is valid or not
- First-Order Predicate Logic
 - Axiomatic formulation is consistent, complete, and semidecidable
 - **Semidecidable**: algorithm determines the truth of any postulated implication, if it terminates. But, termination is guaranteed only if postulate is true

1.3: Strategies for Proofs

- Background
 - Intuition identifies what might be true and why
 - Rigorous proofs verify and communicate that intuition
 - A **proof** is a sequence of verifiable steps from assumptions to conclusion
 - Definitions map between words and symbols (e.g., $P(x) = "x \text{ is even}"$)
- Types of Proofs for $P \rightarrow Q$
 - **Direct**: Assume P true and give steps that lead to Q
 - **Contrapositive**: Proof of the equivalent statement $\neg Q \rightarrow \neg P$
 - **Contradiction**: Using $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$, one supposes that both P and $\neg Q$ are true and then gives steps leading to a contradiction
 - **Induction**: For predicate $P(n)$, prove $Q = "\forall n \in \mathbb{N}, P(n)"$ by establishing the premise $P = "P(1) \wedge (\forall n \in \mathbb{N}, P(n) \rightarrow P(n+1))"$
- Whiteboard Examples
 - Prove $\sqrt{2}$ is irrational by showing contrapositive via contradiction
 - For $P(n) = "\sum_{i=1}^n i = \frac{n^2+n}{2}"$, prove $"\forall n \in \mathbb{N}, P(n)"$ via induction

- To continue studying after this video –
 - Try the suggested reading: Course Notes EF 1.2.3-1.3
 - Or the optional reading: PAF 1.1-2.6
 - Also, look at the problems in Assignment 1