# ECE 586: Vector Space Methods
# Lecture 21: Projection onto Convex Sets

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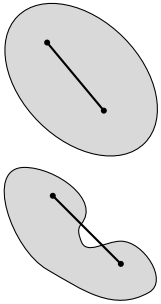
# 5.3: Convexity

Convexity is a useful property defined for sets, spaces, and functionals that simplifies analysis and optimization.



## Definition (convex set)

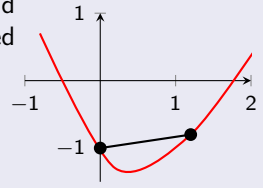
Let $V$ be a vector space over $\mathbb{R}$. The subset $A \subseteq V$ is called a convex set if, for all $\underline{a}_1, \underline{a}_2 \in A$ and $\lambda \in (0,1)$, we have $\lambda \underline{a}_1 + (1-\lambda)\underline{a}_2 \in A$. It is strictly convex if, for all $\underline{a}_1, \underline{a}_2 \in \overline{A}$ and $\lambda \in (0,1)$, $\lambda \underline{a}_1 + (1-\lambda)\underline{a}_2 \in A^\circ$.

## Definition (convex function)

Let $V$ be a vector space, $A \subseteq V$ be a convex set, and $f : V \to \mathbb{R}$ be a functional. The functional $f$ is called convex on $A$ if, for all $\underline{a}_1, \underline{a}_2 \in A$ and $\lambda \in (0,1)$,

$$f(\lambda \underline{a}_1 + (1-\lambda)\underline{a}_2) \leq \lambda f(\underline{a}_1) + (1-\lambda) f(\underline{a}_2).$$

It is strictly convex if equality implies $\underline{a}_1 = \underline{a}_2$.



# 5.3: Convex Optimization

## Definition

Let $(X, \|\cdot\|)$ be a normed vector space. Then, a real functional $f : X \to \mathbb{R}$ achieves a local minimum value at $\underline{x}_0 \in X$ if:

there is an $\epsilon > 0$ such that, for all $\underline{x} \in X$ satisfying $\|\underline{x} - \underline{x}_0\| < \epsilon$, we have $f(\underline{x}) \geq f(\underline{x}_0)$. If this lower bound holds for all $x \in X$, then the local minimum is also a global minimum value.

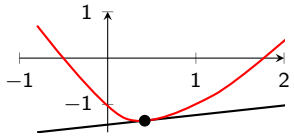
## Theorem

*Let $(X, \|\cdot\|)$ be a normed vector space, $A \subseteq X$ be a convex set, and $f : X \to \mathbb{R}$ be a convex functional on $A$. Then, any local minimum value of $f$ on $A$ is a global minimum value on $A$. If the functional is strictly convex on $A$ and achieves a local minimum value on $A$, then there is a unique point $\underline{x}_0 \in A$ that achieves the global minimum value on $A$.*

Prove in live session and discuss strict convexity of induced norm squared

# 5.3: Convex Optimization and Derivatives

Let $(X, \|\cdot\|)$ be a normed vector space and $f: X \to \mathbb{R}$ be a convex functional on a convex set $A \subseteq X$.




## Theorem

*If $f$ has a directional derivative at $\underline{x}_0 \in A$ in the direction $\underline{x} - \underline{x}_0$, then*

$$f(\underline{x}) \geq f(\underline{x}_0) + \delta f(\underline{x}_0; \underline{x} - \underline{x}_0)$$

*for all $\underline{x} \in A$. If $f$ is strictly convex then the inequality is strict for $\underline{x} \neq \underline{x}_0$.*

## Corollary

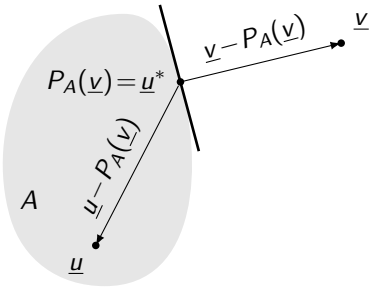
*If $f$ has directional derivatives in all directions at $\underline{x}_0 \in A$ and they all equal zero, then*

$$f(\underline{x}_0) = \min_{\underline{x} \in A} f(\underline{x}).$$

*If $f$ is strictly convex, then $\underline{x}_0$ is the unique minimizer over $A$.*

Prove corollary in live session

# 4.6: Projection onto Convex Sets




The projection of vectors onto subspaces can be generalized to convex sets

For Hilbert space $V$ and closed convex set $A \subseteq V$, let $P_A : V \to A$ denote the orthogonal projection of $\underline{v} \in V$ onto $A$:

$$P_A(\underline{v}) \triangleq \arg\min_{\underline{u} \in A} \|\underline{u} - \underline{v}\|$$

**Theorem**

*The orthogonal projection of $\underline{v} \in V$ onto a closed convex set $A \subseteq V$ exists and is unique.*

**Theorem**

*For any $\underline{v} \notin A$, we have $\underline{u}^* = P_A(\underline{v})$ iff $\langle \underline{v} - \underline{u}^* , \underline{u} - \underline{u}^* \rangle \leq 0$ for all $\underline{u} \in A$.*

Prove theorem in live session for compact $A$ via convex optimization

# Next Steps

- To continue studying after this video –
  - Try the required reading: Course Notes EF 4.6, 5.2 - 5.3
  - Also, look at the problems in Assignments 8 and 9