ECE 586: Vector Space Methods
Lecture 22: Alternating Projection

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Alternating Projection for Subspaces

Let $P_U$ and $P_W$ be orthogonal projections onto closed subspaces $U$ and $W$ of a Hilbert space $V$. For an arbitrary $v_0 \in V$, what is the behavior of the vector sequence $v_n$ generated by alternating projection:

$$v_{n+1} = \begin{cases} P_U v_n & \text{if } n \text{ is even} \\ P_W v_n & \text{if } n \text{ is odd.} \end{cases}$$

Since $P_U v = v$ (resp. $P_W v = v$) if and only if $v \in U$ (resp. $v \in W$), it is easy to see that any vector $v \in U \cap W$ is a fixed point of this recursion.

Letting $P_{U \cap W}$ denote the orthogonal projection onto $U \cap W$, one can show that the sequence $v_n$ converges to $P_{U \cap W} v_0$.

**Theorem**

The sequence $v_n$ converges to $P_{U \cap W} v_0$, its projection onto $U \cap W$. 
The orthogonal projection of $v \in V$ onto a 1D subspace $W = \text{span}(w)$ is

$$P_W(v) = \frac{\langle v, w \rangle}{\|w\|^2}w.$$ 

A subset of $V$ that satisfies a single linear equality of the form $\langle v, w \rangle = 0$ is a subspace $U \subset V$ with co-dimension one (i.e., $\dim(U) = \dim(V) - 1$). Also, $U = W^\perp$ for a 1D subspace $W$ and $U$ is a hyperplane containing $0$. Thus,

$$P_U(v) = P_{W^\perp}(v) = v - \frac{\langle v, w \rangle}{\|w\|^2}w.$$
4.6: Projection onto Hyperplanes

The linear equation $\langle v, w \rangle = c$ defines a shifted subspace $U + v_0$ (for any $v_0 \in V$ satisfying $\langle v_0, w \rangle = c$) with co-dimension one (i.e., a hyperplane):

$$\langle v, w \rangle = \langle u + v_0, w \rangle = \langle u, w \rangle + \langle v_0, w \rangle = 0 + c = c.$$

One can project onto $U + v_0$ by shifting, projecting, and shifting back:

$$P_{U+v_0}(v) = \left(\left(v - v_0\right) - \frac{\langle v - v_0, w \rangle}{\|w\|^2} w\right) + v_0\]

$$= v - \frac{\langle v, w \rangle - c}{\|w\|^2} w,$$
Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ be define a set of $m$ linear equations in $n$ variables with at least one solution.

The goal is to use alternating projection find a solution $x^*$ such that $Ax^* = b$. If $b = 0$, then the set of solutions is a subspace equal to the null space of $A$,

$$
\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\} = \bigcap_{i=1}^{m} \left\{ x \in \mathbb{R}^n \mid \sum_{j=0}^{n} a_{i,j}x_j = b_i = 0 \right\}.
$$

The result follows because $\mathcal{N}(A)$ is the intersection of $m$ hyperplane subspaces (i.e., subspaces of dimension $n - 1$). But, what if $b \neq 0$?
The idea is to iteratively project a candidate vector onto linear equality constraints. For a matrix \( A \in \mathbb{R}^{m \times n} \) and vector \( b \in \mathbb{R}^m \), the algorithm starts from \( v_0 = 0 \) and defines \( v_{i+1} \) to be the projection of \( v_i \) onto the set

\[
W_i = \left\{ v \in \mathbb{R}^n \mid \sum_{k=1}^n a_{\sigma(i),k} v_k = b_{\sigma(i)} \right\},
\]

where \( \sigma(i) = (i \mod m) + 1 \).

Using the previously derived projection formula, this gives

\[
v_{i+1} = (1 - s)v_i + s P_{W_{\sigma(i)}}(v_i) = v_i - s \frac{\langle v_i, a_{\sigma(i)} \rangle - b_{\sigma(i)}}{\|a_{\sigma(i)}\|^2} a_{\sigma(i)},
\]

where \( s \in (0, 1] \) is the step-size and \( a_j \) is the \( j \)-th row of the matrix \( A \).

Note: The true projection uses \( s = 1 \) but \( s < 1 \) may work better if \( b \notin \mathcal{R}(A) \).
Let $C_1, C_2, \ldots, C_m$ be closed convex subsets in a Hilbert space $V$. The **alternating projection algorithm finds a point in their intersection**. Starting from any $v_0 \in V$, the alternating projection algorithm computes

$$v_{i+1} = (1 - s)v_i + s P_{C_{\sigma(i)}}(v_i),$$

where $\sigma(i) = (i \mod m) + 1$ and $s \in (0, 1]$.

**Remark**

The intersection of convex sets is convex, so $C \triangleq \bigcap_{i=1}^m C_i$ is convex set. Ideally, alternating projection would give $v_i \to P_C(v_0)$ but it does not :-(

**Theorem (Bregman)**

*For finite-dimensional $V$, there is some $v \in \bigcap_{i=1}^m C_i$ such that $v_i \to v$.***
4.6: Orthogonal Projection onto Half Spaces

For \( \mathbf{w} \in V \), let \( H = \{ \mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle \geq c \} \) be a closed convex half space.

- For \( \mathbf{v} \in H \), the projection satisfies \( P_H(\mathbf{v}) = \mathbf{v} \)
- For \( \mathbf{v} \notin H \), the projection satisfies \( P_H(\mathbf{v}) = P_{U+\mathbf{v}_0}(\mathbf{v}) \) because the closest point in \( H \) achieves the inequality with equality

It follows that

\[
P_H(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{if } \langle \mathbf{v}, \mathbf{w} \rangle \geq c \\ \mathbf{v} - \frac{\langle \mathbf{v}, \mathbf{w} \rangle - c}{\|\mathbf{w}\|^2} \mathbf{w} & \text{if } \langle \mathbf{v}, \mathbf{w} \rangle < c \end{cases}
\]

Thus, alternating projection can find a feasible vector \( \mathbf{x} \in \mathbb{R}^3 \) satisfying

\[
\begin{align*}
2x_1 - x_2 + x_3 & \geq -1 \\
x_1 + 2x_3 & \geq 2 \\
-7x_1 + 4x_2 - 6x_3 & \geq 1 \\
-3x_1 + x_2 - 2x_3 & \geq 0
\end{align*}
\]
Next Steps

- To continue studying after this video –
  - Try the required reading: Course Notes EF 4.6
  - Look at the Mini-Project Handout on Alternating Projection
  - Also, look at the related problems in Assignments 8 and 9