ECE 586: Vector Space Methods
Lecture 22: Alternating Projection

Henry D. Pfister
Duke University

Alternating Projection for Subspaces

Let P_U and P_W be orthogonal projections onto closed subspaces U and W of a Hilbert space V. For an arbitrary $\underline{v}_0 \in V$, what is the behavior of the vector sequence $\underline{v}_n$ generated by alternating projection:

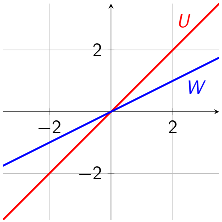
$$\underline{v}_{n+1} = \begin{cases} P_U \underline{v}_n & \text{if } n \text{ is even} \\ P_W \underline{v}_n & \text{if } n \text{ is odd.} \end{cases}$$

Since $P_U \underline{v} = \underline{v}$ (resp. $P_W \underline{v} = \underline{v}$) if and only if $\underline{v} \in U$ (resp. $\underline{v} \in W$), it is easy to see that any vector $\underline{v} \in U \cap W$ is a fixed point of this recursion.

Letting $P_{U\cap W}$ denote the orthogonal projection onto $U \cap W$, one can show that the sequence $\underline{v}_n$ converges to $P_{U\cap W} \underline{v}_0$.

Theorem

The sequence $\underline{v}_n$ converges to $P_{U\cap W} \underline{v}_0$, its projection onto $U \cap W$.



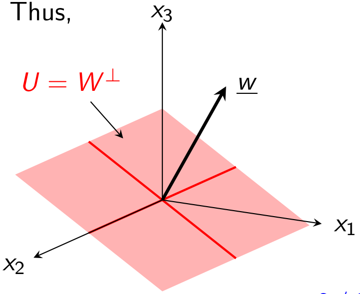
4.6: Projection onto Hyperplane Subspaces

The orthogonal projection of $\underline{v} \in V$ onto a 1D subspace $W = \text{span}(\underline{w})$ is

$$P_W(\underline{v}) = \frac{\langle \underline{v}, \underline{w} \rangle}{\|\underline{w}\|^2} \underline{w}.$$

A subset of V that satisfies a single linear equality of the form $\langle \underline{v}, \underline{w} \rangle = 0$ is a subspace $U \subset V$ with co-dimension one (i.e., $\dim(U) = \dim(V) - 1$). Also, $U = W^\perp$ for a 1D subspace W and U is a hyperplane containing $\underline{0}$. Thus,

$$P_U(\underline{v}) = P_{W^\perp}(\underline{v}) = \underline{v} - \frac{\langle \underline{v}, \underline{w} \rangle}{\|\underline{w}\|^2} \underline{w}.$$

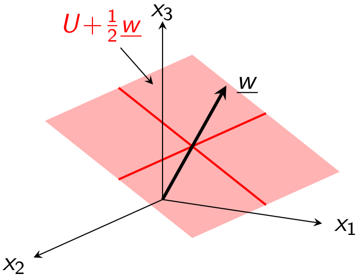


The linear equation $\langle \underline{v}, \underline{w} \rangle = c$ defines a shifted subspace $U + \underline{v}_0$ (for any $\underline{v}_0 \in V$ satisfying $\langle \underline{v}_0, \underline{w} \rangle = c$) with co-dimension one (i.e., a hyperplane):

$$\langle \underline{v}, \underline{w} \rangle = \langle \underline{u} + \underline{v}_0, \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}_0, \underline{w} \rangle = 0 + c = c.$$

One can project onto $U + \underline{v}_0$ by shifting, projecting, and shifting back:

$$\begin{aligned} P_{U+\underline{v}_0}(\underline{v}) &= \left((\underline{v} - \underline{v}_0) - \frac{\langle \underline{v} - \underline{v}_0, \underline{w} \rangle}{\|\underline{w}\|^2} \underline{w} \right) + \underline{v}_0 \\ &= \underline{v} - \frac{\langle \underline{v}, \underline{w} \rangle - c}{\|\underline{w}\|^2} \underline{w}, \end{aligned}$$



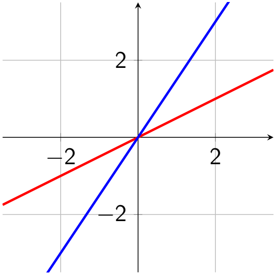
4.6: Solving Linear Equations via Alternating Projection

Let $A \in \mathbb{R}^{m \times n}$ and $\underline{b} \in \mathbb{R}^m$ be define a set of m linear equations in n variables with at least one solution.

The goal is to use alternating projection find a solution $\underline{x}^*$ such that $A\underline{x}^* = \underline{b}$. If $\underline{b} = \underline{0}$, then the set of solutions is a subspace equal to the null space of A,

$$\mathcal{N}(A) = \{\underline{x} \in \mathbb{R}^n \,|\, A\underline{x} = \underline{0}\} = \bigcap_{i=1}^{m} \left\{ \underline{x} \in \mathbb{R}^n \,\middle|\, \textstyle\sum_{j=0}^{n} a_{i,j} x_j = b_i = 0 \right\}.$$

The result follows because $\mathcal{N}(A)$ is the intersection of m hyperplane subspaces (i.e., subspaces of dimension $n-1$). But, what if $\underline{b} \neq \underline{0}$?



4.6: Kaczmarz's Algorithm

The idea is to iteratively project a candidate vector onto linear equality constraints. For a matrix $A \in \mathbb{R}^{m \times n}$ and vector $\underline{b} \in \mathbb{R}^m$, the algorithm starts from $\underline{v}_0 = \underline{0}$ and defines $\underline{v}_{i+1}$ to be the projection of $\underline{v}_i$ onto the set

$$W_i = \left\{ \underline{v} \in \mathbb{R}^n \,\middle|\, \sum_{k=1}^{n} a_{\sigma(i),k} v_k = b_{\sigma(i)} \right\},$$

where $\sigma(i) = (i \bmod m) + 1$.

Using the previously derived projection formula, this gives

$$\underline{v}_{i+1} = (1-s)\underline{v}_i + s\, P_{W_{\sigma(i)}}(\underline{v}_i) = \underline{v}_i - s\frac{\langle \underline{v}_i, \underline{a}_{\sigma(i)} \rangle - b_{\sigma(i)}}{\|\underline{a}_{\sigma(i)}\|^2} \underline{a}_{\sigma(i)},$$

where $s \in (0, 1]$ is the step-size and $\underline{a}_j$ is the j-th row of the matrix A.

Note: The true projection uses $s = 1$ but $s < 1$ may work better if $\underline{b} \notin \mathcal{R}(A)$.

4.6: Alternating Projection onto Convex Sets

Let $C_1, C_2, \ldots, C_m$ be closed convex subsets in a Hilbert space V. The alternating projection algorithm finds a point in their intersection. Starting from any $\underline{v}_0 \in V$, the alternating projection algorithm computes

$$\underline{v}_{i+1} = (1-s)\underline{v}_i + s\, P_{C_{\sigma(i)}}(\underline{v}_i),$$

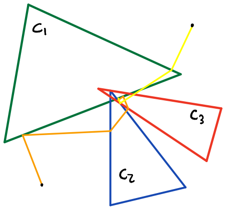
where $\sigma(i) = (i \bmod m) + 1$ and $s \in (0, 1]$.

Remark

The intersection of convex sets is convex, so $C \triangleq \cap_{i=1}^m C_i$ is convex set. Ideally, alternating projection would give $\underline{v}_i \to P_C(\underline{v}_0)$ but it does not :-(

Theorem (Bregman)

For finite-dimensional V, there is some $\underline{v} \in \cap_{i=1}^m C_i$ such that $\underline{v}_i \to \underline{v}$.



4.6: Orthogonal Projection onto Half Spaces

For $\underline{w} \in V$, let $H = \{\underline{v} \in V \mid \langle \underline{v}, \underline{w} \rangle \geq c\}$ be a closed convex half space.

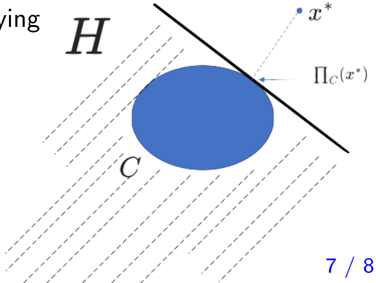
- For $\underline{v} \in H$, the projection satisfies $P_H(\underline{v}) = \underline{v}$
- For $\underline{v} \notin H$, the projection satisfies $P_H(\underline{v}) = P_{U+\underline{v}_0}(\underline{v})$ because the closest point in H achieves the inequality with equality

It follows that

$$P_H(\underline{v}) = \begin{cases} \underline{v} & \text{if } \langle \underline{v}, \underline{w} \rangle \geq c \\ \underline{v} - \frac{\langle \underline{v}, \underline{w} \rangle - c}{\|\underline{w}\|^2} \underline{w} & \text{if } \langle \underline{v}, \underline{w} \rangle < c \end{cases}$$

Thus, alternating projection can find a feasible vector $\underline{x} \in \mathbb{R}^3$ satisfying

$$\begin{aligned} 2x_1 - x_2 + x_3 &\geq -1 \\ x_1 + 2x_3 &\geq 2 \\ -7x_1 + 4x_2 - 6x_3 &\geq 1 \\ -3x_1 + x_2 - 2x_3 &\geq 0 \end{aligned}$$



Next Steps

- To continue studying after this video –
  - Try the required reading: Course Notes EF 4.6
  - Look at the Mini-Project Handout on Alternating Projection
  - Also, look at the related problems in Assignments 8 and 9