Alternate Projection for Subspaces

Let $P_U$ and $P_W$ be orthogonal projections onto closed subspaces $U$ and $W$ of a Hilbert space $V$. For an arbitrary $v_0 \in V$, what is the behavior of the vector sequence $v_n$ generated by alternating projection:

\[ v_{n+1} = \begin{cases} 
P_U v_n & \text{if } n \text{ is even} \\
P_W v_n & \text{if } n \text{ is odd.}
\end{cases} \]

Since $P_U v = v$ (resp. $P_W v = v$) if and only if $v \in U$ (resp. $v \in W$), it is easy to see that any vector $v \in U \cap W$ is a fixed point of this recursion.

Letting $P_{U \cap W}$ denote the orthogonal projection onto $U \cap W$, one can show that the sequence $v_n$ converges to $P_{U \cap W} v_0$.

**Theorem**

The sequence $v_n$ converges to $P_{U \cap W} v_0$, its projection onto $U \cap W$. 
4.6: Projection onto 1D Subspaces

The orthogonal projection of \( \mathbf{v} \in V \) onto a 1D subspace \( W = \text{span}(\mathbf{w}) \) is

\[
P_W(\mathbf{v}) = \frac{\langle \mathbf{v} | \mathbf{w} \rangle}{\| \mathbf{w} \|^2} \mathbf{w}.
\]

A subspace \( U \subset V \) with co-dimension one (i.e., \( \dim(U) = \dim(V) - 1 \)) is a subset of \( V \) that satisfies a single linear equality of the form \( \langle \mathbf{v} | \mathbf{w} \rangle = 0 \). Thus, \( U = W^\perp \) for a 1D subspace \( W \) and

\[
P_U(\mathbf{v}) = P_{W^\perp}(\mathbf{v}) = \mathbf{v} - \frac{\langle \mathbf{v} | \mathbf{w} \rangle}{\| \mathbf{w} \|^2} \mathbf{w}.
\]
From the linear equality $\langle v | w \rangle = c$, one gets a shifted subspace $U + v_0$ ($v_0$ is any vector in $V$ satisfying $\langle v_0 | w \rangle = c$) with co-dimension one:

$\langle v | w \rangle = \langle u + v_0 | w \rangle = \langle u | w \rangle + \langle v_0 | w \rangle = 0 + c = c$.

One can project onto $U + v_0$ by shifting, projecting, and shifting back:

$$P_{U + v_0}(v) = \left( v - v_0 - \frac{\langle v - v_0 | w \rangle}{\| w \|^2} w \right) + v_0 = v - \frac{\langle v | w \rangle - c}{\| w \|^2} w,$$
Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ be define a set of $m$ linear equations in $n$ variables with at least one solution.

The goal is to use alternating projection find a solution $x^*$ such that $Ax^* = b$. If $b = 0$, then the set of solutions is a subspace equal to the null space of $A$,

$$
\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\} = \bigcap_{i=1}^m \left\{x \in \mathbb{R}^n \mid \sum_{j=0}^n a_{i,j}x_j = b_i = 0\right\}.
$$

The result follows because $\mathcal{N}(A)$ is the intersection of $m$ subspaces of dimension $n - 1$. But, what if $b \neq 0$?
4.6: Kaczmarz’s Algorithm

The idea is to iteratively project a candidate vector onto linear equality constraints. For a matrix $A \in \mathbb{R}^{m \times n}$ and vector $b \in \mathbb{R}^m$, the algorithm starts from $v_0 = 0$ and defines $v_{i+1}$ to be the projection of $v_i$ onto the set

$$W_i = \left\{ v \in \mathbb{R}^n \left| \sum_{k=1}^{n} a_{\sigma(i),k} v_k = b_{\sigma(i)} \right. \right\},$$

where $\sigma(i) = (i \mod m) + 1$.

Using the previously derived projection formula, this gives

$$v_{i+1} = (1 - s)v_i + s P_{W_{\sigma(i)}}(v_i) = v_i - s \frac{\langle v_i, a_{\sigma(i)} \rangle - b_{\sigma(i)}}{\|a_{\sigma(i)}\|_2^2} a_{\sigma(i)},$$

where $s \in (0, 1]$ is the step-size and $a_j$ is the $j$-th row of the matrix $A$.

Note: The true projection uses $s = 1$ but $s < 1$ may work better if $b \notin \mathcal{R}(A)$. 
Let $C_1, C_2, \ldots, C_m$ be closed convex subsets in a Hilbert space $V$. The alternating projection algorithm finds a point in their intersection. Starting from any $v_0 \in V$, the alternating projection algorithm computes

$$v_{i+1} = (1 - s)v_i + s P_{C_{\sigma(i)}}(v_i),$$

where $\sigma(i) = (i \mod m) + 1$ and $s \in (0, 1]$.

**Remark**

The intersection of convex sets is convex, so $C \triangleq \bigcap_{i=1}^m C_i$ is convex set. Ideally, alternating projection would give $v_i \to P_C(v_0)$ but it does not :-(

**Theorem (Bregman)**

For finite-dimensional $V$, there is some $v \in \bigcap_{i=1}^m C_i$ such that $v_i \to v$. 
4.6: Orthogonal Projection onto Half Spaces

For $\mathbf{w} \in V$, let $H = \{ \mathbf{v} \in V \mid \langle \mathbf{v} | \mathbf{w} \rangle \geq c \}$ be a closed convex half space.

- For $\mathbf{v} \in H$, the projection satisfies $P_H(\mathbf{v}) = \mathbf{v}$
- For $\mathbf{v} \notin H$, the projection satisfies $P_H(\mathbf{v}) = P_{U+\mathbf{v}_0}(\mathbf{v})$ because the closest point in $H$ achieves the inequality with equality

It follows that

$$P_H(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{if } \langle \mathbf{v} | \mathbf{w} \rangle \geq c \\ \mathbf{v} - \frac{\langle \mathbf{v} | \mathbf{w} \rangle - c}{\|\mathbf{w}\|^2} \mathbf{w} & \text{if } \langle \mathbf{v} | \mathbf{w} \rangle < c \end{cases}$$

Thus, alternating projection can find a feasible vector $\mathbf{x} \in \mathbb{R}^3$ satisfying

$$2x_1 - x_2 + x_3 \geq -1$$
$$x_1 + 2x_3 \geq 2$$
$$-7x_1 + 4x_2 - 6x_3 \geq 1$$
$$-3x_1 + x_2 - 2x_3 \geq 0$$
Next Steps

To continue studying after this video –

- Try the required reading: Course Notes EF 4.6
- Look at the Mini-Project Handout on Alternating Projection
- Also, look at the related problems in Assignments 8 and 9