ECE 586: Vector Space Methods
Lecture 3 Flip Video: Set Theory

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1.4: Set Theory

- is the foundation (along with logic) of all modern mathematics
  - Numbers, relations, functions, ... are all defined using set theory
  - But, it's not as simple as one would hope because naive approaches are inconsistent (i.e., for some P, the definitions imply P and $\neg P$)
  - Axiomatic approaches avoid contradictions but are overly complicated
  - Thus, we use naive set theory, which defines the operations of set theory without worrying about paradoxes. This is sufficient for engineering math.
- Naive Set Theory
  - Set defined as "any collection of objects, mathematical or otherwise"
  - Ex. Consider "the set of all books published in 2007"
  - Objects in a set are called elements or members of the set
  - Logical statement: "a is a member of the set A" is written $a \in A$
  - Its negation: "a is not a member of the set A" is written $a \notin A$

1.4: Using Set Theory

- Defining Sets
    - One can present a set by listing elements: standard English vowels
    $$A = \{a, e, i, o, u\}$$
    - Element order is irrelevant: $\{i, o, u, a, e\}$ is the same as A
    Repeated elements have no effect: $\{a, e, i, o, u, e, o\}$ same as A
    - A singleton is a set containing exactly one element such as $\{a\}$
    - Standard sets: Integers $\mathbb{Z}$, Real numbers $\mathbb{R}$, and Complex numbers $\mathbb{C}$
- Building new sets from old
    - Set-builder notation: For logical predicate $P(x)$ defined on $x \in X$,
    "A is the set of elements in X such that $P(x)$ is true" is denoted by
    $$A = \{x \in X | P(x)\}$$
    - If no $x \in X$ satisfies $P(x)$, then result is the empty set $\emptyset$
    - Ex. natural numbers $\mathbb{N}$ and positive prime integers P:
    $$\mathbb{N} = \{x \in \mathbb{Z} | x \geq 1\} = \{1, 2, 3, 4, \ldots\}$$
    $$P = \{x \in \mathbb{Z} | x \geq 1 \text{ and } \text{“}x \text{ is prime”}\} = \{2, 3, 5, 7, 11, \ldots\}$$

1.4: Set Properties

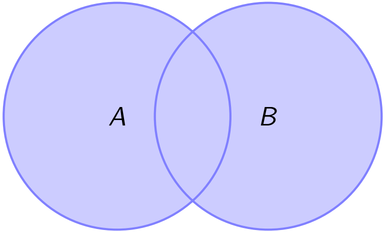
- Cardinality
  - For set A, the cardinality $|A|$ is the number of elements in A

$$|\{a, e, i, o, u\}| = 5$$

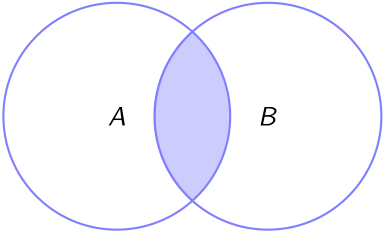
  - If there is a one-to-one correspondence between A and the natural numbers $\mathbb{N}$, then A is called countably infinite and $|A| = \infty$
  - Ex. Set of rational numbers $\mathbb{Q} \triangleq \{q \in \mathbb{R} \mid \exists n \in \mathbb{N}, nq \in \mathbb{Z}\}$ is countably infinite (e.g., list rationals m/n with $|m| \leq n^2$)
  - If $|A| = \infty$ but not countably infinite, then A is uncountably infinite
  - Ex. Real numbers are uncountably infinite by Cantor's diagonal argument (not covered in this class but worth googling if you haven't seen it)

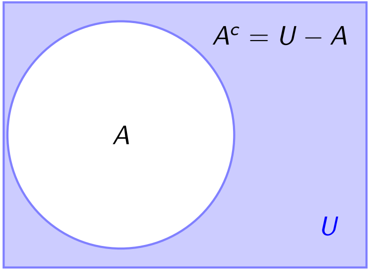
1.4: Venn Diagrams

$A \cup B$

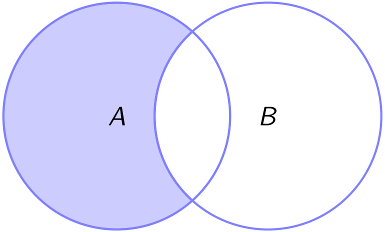


$A \cap B$





$A - B$



1.4: From Logic to Set Theory

- Operations on sets A, B
  - Union of A and B ($A \cup B$): set of elements in either A or B
    $$x \in A \cup B \Leftrightarrow (x \in A) \vee (x \in B)$$
  - Intersection of A and B ($A \cap B$): set of elements in both A and B
    $$x \in A \cap B \Leftrightarrow (x \in A) \wedge (x \in B)$$
  - Set difference $A - B$ (or $A \setminus B$): set of elements in A but not in B
    $$x \in A - B \Leftrightarrow (x \in A) \wedge (x \notin B)$$
  - Complement A^c for implied universal set U is defined by $A^c = U - A$
    $$x \in A^c \Leftrightarrow x \notin A$$

- Relationships between sets A, B
  - A equals B (denoted $A = B$) if both sets have the same elements
    $$\text{“}A = B\text{”} \Leftrightarrow \forall x\,((x \in A) \leftrightarrow (x \in B))$$
  - A is a subset of B (denoted $A \subseteq B$) if all elements in A are also in B
    $$\text{“}A \subseteq B\text{”} \Leftrightarrow \forall x\,((x \in A) \to (x \in B))$$
  - A is a proper subset of B (denoted $A \subset B$) if $A \subseteq B$ and $A \neq B$
  - Two sets are called disjoint if $A \cap B = \emptyset$

1.4: De Morgan, Infinite Operations, and Negation

- De Morgan's identity $\neg(P \vee Q) = \neg P \wedge \neg Q$ with $P/Q =$ "$x \in A/B$" implies:

$$(A \cup B)^c = A^c \cap B^c$$

- Infinite unions and intersections are defined by:

$$\bigcup_{\alpha \in I} S_\alpha \triangleq \{x | x \in S_\alpha \text{ for some } \alpha \in I\}$$
$$\bigcap_{\alpha \in I} S_\alpha \triangleq \{x | x \in S_\alpha \text{ for all } \alpha \in I\}$$

- De Morgan's identity still applies:

$$\text{“}x \in \bigcup_{\alpha \in I} S_\alpha\text{”} \Leftrightarrow \text{“}\exists \alpha \in I, x \in S_\alpha\text{”}$$
$$\text{“}x \in \left(\bigcup_{\alpha \in I} S_\alpha\right)^c\text{”} \Leftrightarrow \text{“}\forall \alpha \in I, x \notin S_\alpha\text{”} \Leftrightarrow \text{“}x \in \bigcap_{\alpha \in I} S_\alpha^c\text{”}$$

Example (Russell's Paradox)

Let $R = \{S | S \notin S\}$ be the set of all sets that do not contain themselves. This set exists in naive set theory (though it may empty) simply because it is described by the above sentence. The paradox arises from the fact that this definition leads to the logical contradiction: $R \notin R \leftrightarrow R \in R$.

- Sets that contain themselves in naive set theory?
- What does Russell's paradox show?
  - It shows that naive set theory is not consistent because it allows constructions leading to contradictions
  - It is avoided in axiomatic formulations by restricting constructions
  - It also implies that R cannot exist in any consistent set theory

Next Steps

- To continue studying after this video –
  - Try the suggested reading: Course Notes EF 1.4
  - Or the optional reading: PAF 3.1-5.3
  - Also, look at the problems in Assignment 2