ECE 586: Vector Space Methods
Lecture 4 Flip Video: Relations and Functions

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1.4: Cartesian Products and Abstract Relations

- Sets of tuples and vectors
  - For sets A, B, the Cartesian product $A \times B$ is the set of ordered pairs

$$A \times B \triangleq \{(a, b) \mid a \in A, b \in B\}$$

  - For n-tuples from the same set, we write $A^n = A \times A \times \cdots \times A$
  - Ex. For $A = \{a, b\}$ and $B = \{c, d\}$, we have

$$A \times B = \{(a, c), (a, d), (b, c), (b, d)\}$$
$$A^2 = A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$$
$$A^3 = \{(a, a, a), (a, a, b), (a, b, a), (a, b, b), (b, a, a), (b, a, b), (b, b, a), (b, b, b)\}$$

Definition

A relation $\sim$ between elements of A is defined by the subset $R \subseteq A \times A$. Specifically, we say the relation $x \sim y$ holds if and only if $(x, y) \in R$.

Relations are the abstraction of operators like $\{=, \leq, \geq, <, >\}$

1.4: Properties of Relations

- The relation $\sim$ on A is said to be:
  - Reflexive if $x \sim x$ holds for all $x \in A$ (i.e., $\forall x \in A, (x, x) \in R$)
  - Symmetric if $x \sim y$ then $y \sim x$ for all $x, y \in A$
  - Transitive if $x \sim y$ and $y \sim z$, then $x \sim z$ for all $x, y, z \in A$

- It is an equivalence relation if it is reflexive, symmetric, and transitive
  - Ex. let A be a set of people and $P(x, y)$ be the statement: "x has the same birthday (month and day) as y"
  - Define relation $\sim$ such that $x \sim y$ holds iff $P(x, y)$ is true. Then,

$$R = \{(x, y) \in A \times A \,|\, P(x, y)\}$$

  - Partitions A into disjoint equivalence classes: $[a] \triangleq \{x \in A | x \sim a\}$
  - Ex. $\sim$ has an equivalence class for each possible day in a year
  - Set of equivalence classes called the quotient set: $A \backslash \sim \triangleq \{[a] | a \in A\}$

1.5: Functions

Definition

A function $f\colon X \to Y$ from X to Y is defined by a subset $F \subset X \times Y$ such that $A_x = \{y \in Y | (x, y) \in F\}$ has exactly one element for each $x \in X$. The value of f at $x \in X$, denoted $f(x)$, is unique element in A_x.

- Unpacking the definition
  - Function $f\colon X \to Y$ assigns one value $f(x) \in Y$ to each $x \in X$
  - Notation $f\colon X \to Y$ emphasizes the domain X and the codomain Y
  - Range of f is subset of Y achieved by f, $\{y \in Y \mid \exists x \in X, y = f(x)\}$
  - Since term codomain is uncommon, people sometimes use the term range instead of codomain either intentionally or unintentionally
- In basic math, functions are often described by graphs and formulas
  - This leads students to picture only "nice" functions
  - Ex. Cauchy published incorrect proof of false assertion: "a sequence of continuous functions converging everywhere has a continuous limit"

1.5: Properties of Functions

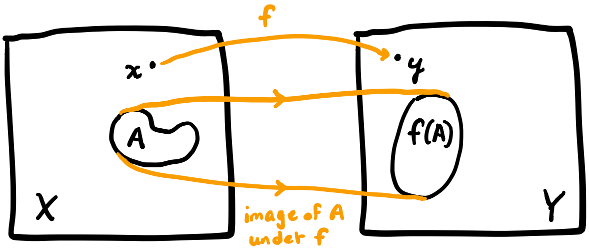
- Equality
  - Two functions are equal if they have the same domain, codomain, and value for all elements of the domain
- A function $f : X \to Y$ is called:
  - one-to-one or injective if, $\forall x, x' \in X$, if $f(x) = f(x')$ then $x = x'$
  - onto or surjective if its range $\{f(x) | x \in X\}$ equals Y
  - one-to-one correspondence or bijective if both one-to-one and onto
- Inversion
  - A bijective function has a unique inverse function $f^{-1} : Y \to X$ satisfying: $\forall x \in X, f^{-1}(f(x)) = x$ and $\forall y \in Y, f(f^{-1}(y)) = y$
  - Any one-to-one function $f : X \to Y$ defines a bijective function $g : X \to R$ where R is the range of f and $g(x) = f(x)$ for all $x \in X$

1.5: Applying Functions to Sets (1)

Definition

For $f : X \to Y$ and subset $A \subseteq X$, the image of A under f is

$$f(A) \triangleq \{y \in Y | \exists x \in A \text{ s.t. } f(x) = y\} = \{f(x) | x \in A\}.$$



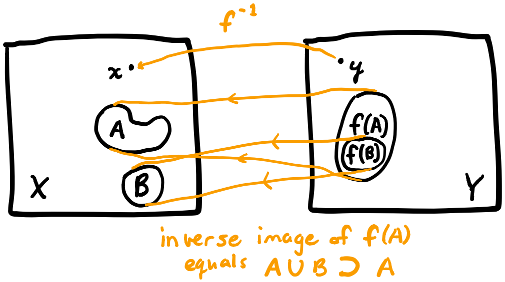
- This implies that the range of f is simply $f(X)$

1.5: Applying Functions to Sets (2)

Definition

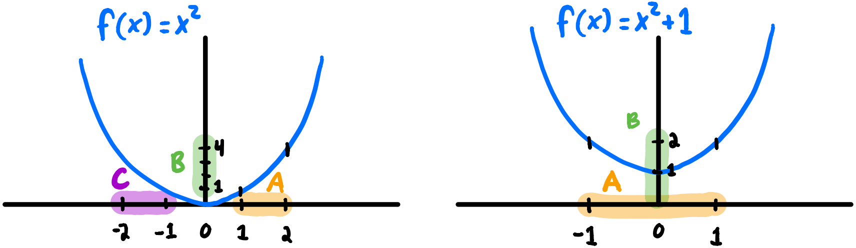
The inverse image or preimage of a subset $B \subseteq Y$ is

$$f^{-1}(B) \triangleq \{x \in X | f(x) \in B\}.$$



- Allowing set-valued images means set-valued inverse always exists
- For a one-to-one f, inverse image of singleton $\{f(x)\}$ is singleton $\{x\}$:

$$f^{-1}(\{f(x)\}) = \{x\}$$



- In general, one can show: $f^{-1}(f(A)) \supseteq A$ and $f(f^{-1}(B)) \subseteq B$
- Left: For $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, let $A = [1, 2]$ and notice that $B = f(A) = [1, 4]$. But, $f^{-1}(B) = [-2, -1] \cup [1, 2] \supset A$
- Right: For $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$, let $B = [0, 2]$ and notice that $A = f^{-1}(B) = [-1, 1]$. But, $f(A) = f([-1, 1]) = [1, 2] \subset B$

Next Steps

- To continue studying after this video –
  - Try the suggested reading: Course Notes EF 1.5
  - Or the optional reading: PAF 4.1-5.3
  - And also looking at problems in Assignment 2