

# ECE 586: Vector Space Methods

## Lecture 4 Flip Video: Relations and Functions

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## 1.4: Cartesian Products and Abstract Relations

- Sets of tuples and vectors
  - For sets  $A, B$ , the **Cartesian product**  $A \times B$  is the set of ordered pairs

$$A \times B \triangleq \{(a, b) \mid a \in A, b \in B\}$$

- For  $n$ -tuples from the same set, we write  $A^n = A \times A \times \cdots \times A$
- Ex. For  $A = \{a, b\}$  and  $B = \{c, d\}$ , we have

$$A \times B = \{(a, c), (a, d), (b, c), (b, d)\}$$

$$A^2 = A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$$

$$A^3 = \{(a, a, a), (a, a, b), (a, b, a), (a, b, b), (b, a, a), (b, a, b), (b, b, a), (b, b, b)\}$$

### Definition

A **relation**  $\sim$  between elements of  $A$  is defined by the subset  $R \subseteq A \times A$ . Specifically, we say the relation  $x \sim y$  holds if and only if  $(x, y) \in R$ .

Relations are the abstraction of operators like  $\{=, \leq, \geq, <, >\}$

## 1.4: Properties of Relations

- The relation  $\sim$  on  $A$  is said to be:
  - **Reflexive** if  $x \sim x$  holds for all  $x \in A$  (i.e.,  $\forall x \in A, (x, x) \in R$ )
  - **Symmetric** if  $x \sim y$  then  $y \sim x$  for all  $x, y \in A$
  - **Transitive** if  $x \sim y$  and  $y \sim z$ , then  $x \sim z$  for all  $x, y, z \in A$
- It is an **equivalence relation** if it is reflexive, symmetric, and transitive
  - Ex. let  $A$  be a set of people and  $P(x, y)$  be the statement:  
"x has the same birthday (month and day) as y"
  - Define **relation**  $\sim$  such that  $x \sim y$  holds iff  $P(x, y)$  is true. Then,
$$R = \{(x, y) \in A \times A \mid P(x, y)\}$$
  - Partitions  $A$  into disjoint **equivalence classes**:  $[a] \triangleq \{x \in A \mid x \sim a\}$
  - Ex.  $\sim$  has an equivalence class for each possible day in a year
  - Set of equivalence classes called the **quotient set**:  $A/\sim \triangleq \{[a] \mid a \in A\}$

### Definition

A **function**  $f: X \rightarrow Y$  from  $X$  to  $Y$  is defined by a subset  $F \subset X \times Y$  such that  $A_x = \{y \in Y \mid (x, y) \in F\}$  has exactly one element for each  $x \in X$ . The **value** of  $f$  at  $x \in X$ , **denoted**  $f(x)$ , is unique element in  $A_x$ .

- Unpacking the definition
  - Function  $f: X \rightarrow Y$  assigns one value  $f(x) \in Y$  to each  $x \in X$
  - Notation  $f: X \rightarrow Y$  emphasizes the **domain**  $X$  and the **codomain**  $Y$
  - **Range** of  $f$  is subset of  $Y$  achieved by  $f$ ,  $\{y \in Y \mid \exists x \in X, y = f(x)\}$
  - Since the term codomain is uncommon, people sometimes use the term range instead of codomain either intentionally or unintentionally
- In basic math, functions are often described by graphs and formulas
  - This leads students to picture only “nice” functions
  - Ex. Cauchy published **incorrect proof of the false assertion**: “a sequence of continuous functions converging everywhere has a continuous limit”

## 1.5: Properties of Functions

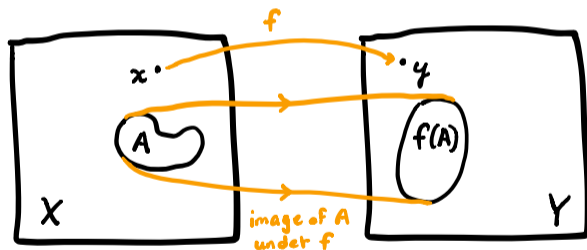
- Equality
  - Two functions are **equal** if they have the same domain, codomain, and value for all elements of the domain
- A function  $f: X \rightarrow Y$  is called:
  - **one-to-one** or **injective** if,  $\forall x, x' \in X$ , if  $f(x) = f(x')$  then  $x = x'$
  - **onto** or **surjective** if its range  $\{f(x) | x \in X\}$  equals  $Y$
  - **one-to-one correspondence** or **bijective** if both one-to-one and onto
- Inversion
  - A bijective function has a unique **inverse function**  $f^{-1}: Y \rightarrow X$  satisfying:  
 $\forall x \in X, f^{-1}(f(x)) = x$  and  $\forall y \in Y, f(f^{-1}(y)) = y$
  - Any one-to-one function  $f: X \rightarrow Y$  defines a bijective function  $g: X \rightarrow R$  where  $R$  is the range of  $f$  and  $g(x) = f(x)$  for all  $x \in X$

## 1.5: Applying Functions to Sets (1)

### Definition

For  $f: X \rightarrow Y$  and subset  $A \subseteq X$ , the **image** of  $A$  under  $f$  is

$$f(A) \triangleq \{y \in Y \mid \exists x \in A \text{ s.t. } f(x) = y\} = \{f(x) \mid x \in A\}.$$



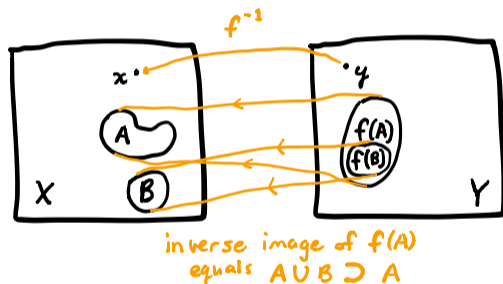
- This implies that the range of  $f$  is simply  $f(X)$

## 1.5: Applying Functions to Sets (2)

### Definition

The **inverse image** or **preimage** of a subset  $C \subseteq Y$  is

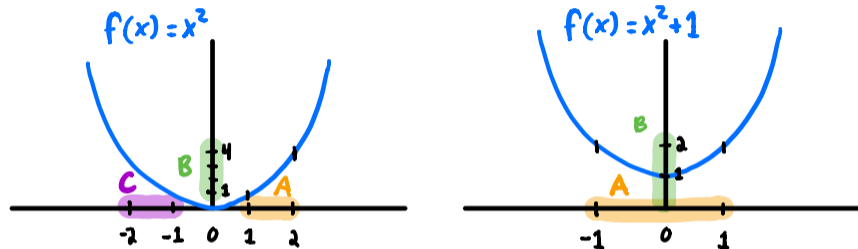
$$f^{-1}(C) \triangleq \{x \in X \mid f(x) \in C\}.$$



- Allowing set-valued images means the **set-valued inverse always exists**
- For a one-to-one  $f$ , the inverse image of a singleton  $\{f(x)\}$  is a singleton:

$$f^{-1}(\{f(x)\}) = \{x\}$$

## 1.5: Applying Functions to Sets (3)



- In general, one can show:  $f^{-1}(f(A)) \supseteq A$  and  $f(f^{-1}(B)) \subseteq B$
- Left: For  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = x^2$ , let  $A = [1, 2]$  and notice that  $B = f(A) = [1, 4]$ . But,  $f^{-1}(B) = [-2, -1] \cup [1, 2] \supset A$
- Right: For  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = x^2 + 1$ , let  $B = [0, 2]$  and notice that  $A = f^{-1}(B) = [-1, 1]$ . But,  $f(A) = f([-1, 1]) = [1, 2] \subseteq B$

- To continue studying after this video –
  - Try the suggested reading: Course Notes EF 1.5
  - Or the optional reading: PAF 4.1-5.3
  - And also looking at problems in Assignment 2