ECE 586: Vector Space Methods
Lecture 5 Flip Video: Metric Spaces and Topology

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2.1: Introduction

What is topology and why do we study it?

- Study of geometric properties preserved by continuous deformations
- Why? Engineers approximate real things by mathematical objects
- Q1: Can a matrix $A$ be approximated closely by a lower rank matrix?
- Q2: Can a function $f(x)$ be approximated well by a degree-2 polynomial?
- In engineering, a topology is typically defined using a metric

Metric Spaces

- A metric space $(X, d)$ is a set $X$ along with a metric $d(x, y)$
- $d(x, y)$ is called the distance between the points $x$ and $y$
- A metric on a set $X$ is a function $d: X \times X \rightarrow \mathbb{R}$ such that:
  1. $d(x, y) \geq 0 \quad \forall x, y \in X$; with equality iff $x = y$ (non-negativity)
  2. $d(x, y) = d(y, x) \quad \forall x, y \in X$ (symmetry)
  3. $d(x, y) + d(y, z) \geq d(x, z) \quad \forall x, y, z \in X$ (triangle inequality)
2.1: Standard Examples of Metric Spaces

- Real numbers \( X = \mathbb{R} \) with **absolute distance** \( d(x, y) = |x - y| \)
  - Foundation for real analysis: 1. and 2. are easy; 3. follows from
  \[
  |x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z|
  \]

- \( X = \mathbb{R}^n \) with **Euclidean metric** \( d(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2} \)
  - Properties 1. and 2. are easy; 3. is a bit harder and will be shown later

- Continuous functions \( f: [a, b] \to \mathbb{R} \) with metric \( d(f, g) = \max_{x \in [a, b]} |f(x) - g(x)| \)
  - Properties 1. and 2. **inherited** from absolute distance; 3. follows from
  \[
  \max_{x \in [a, b]} |f(x) - h(x)| = \max_{x \in [a, b]} |f(x) - g(x) + g(x) - h(x)|
  \leq \max_{x \in [a, b]} [|f(x) - g(x)| + |g(x) - h(x)|]
  \leq \max_{x \in [a, b]} |f(x) - g(x)| + \max_{x \in [a, b]} |g(x) - h(x)| = d(f, g) + d(g, h)
  \]
2.1: Important Concepts in Metric Spaces

- “Set of points with distance less than \( \epsilon \) from a point \( x \)”
  - This is called the open ball of radius \( \epsilon \) centered at \( x \) and given by
    \[
    B_d(x, \epsilon) \triangleq \{ y \in X | d(x, y) < \epsilon \}
    \]
  - \( P = "\text{For all } a \in B_d(x, \epsilon), \text{there is } \delta > 0 \text{ s.t. } B_d(a, \delta) \subseteq B_d(x, \epsilon)" \)

- “Infinite list \( x_1, x_2, x_3, \ldots \) of points in \( X \)”
  - A sequence \( x_i \in X \) for \( i \in \mathbb{N} \) equivalent to \( x_i = f(i) \) for \( f : \mathbb{N} \to X \)
  - Ex. For \( X = \mathbb{R} \) and \( d(x, y) = |x - y| \), let \( x_n = (1 + \frac{1}{n})^n \) for \( n \in \mathbb{N} \)

- “A sequence of points that approaches another point”
  - A sequence \( x_n \) converges to \( x \in X \) (denoted \( x_n \rightarrow x \)) if, for any \( \epsilon > 0 \), there is natural number \( M \) such that \( d(x, x_n) < \epsilon \) for all \( n > M \)
A sequence $x_1, x_2, \ldots$ in $(X, d)$ is a **Cauchy sequence** if, for any $\epsilon > 0$, there is a natural number $N$ (which may depend on $\epsilon$) such that, for all $m, n > N$,

$$d(x_m, x_n) < \epsilon$$

**Theorem:** Every convergent sequence is a Cauchy sequence

- Proof will be given in the lecture

**Converse?** No, there is a counterexample

- Metric space $(X, d)$ with rationals $X = \mathbb{Q}$ and $d(x, y) = |x - y|
- Sequence $x_1 = 2$ and $x_{n+1} = f(x_n) \triangleq \frac{1}{2} x_n + 1/x_n \in \mathbb{Q}$
- One can show: $x_n$ is a Cauchy sequence and $|x_n - \sqrt{2}| \to 0$

- But, according to the definitions, $x_n$ does not converge!
  - Because convergence requires that the limit is in $X$ but $\sqrt{2} \notin \mathbb{Q}$
2.1.1: Metric Topology

**Definition**

Let $W$ be a subset of a metric space $(X, d)$. The set $W$ is called open if, for every $w_0 \in W$, there is an $\epsilon > 0$ such that $B_d(w_0, \epsilon) \subseteq W$.

**Definition**

A subset $W$ of $(X, d)$ is closed if its complement $W^c = X - W$ is open.

**Theorem**

1. $\emptyset$ and $X$ are open sets
2. Any union of open sets is open
3. Any finite intersection of open sets is open
For a metric space \((X, d)\) and subset \(W \subseteq X\):

**Definition**
A point \(w_0 \in W\) is in the **interior** of \(W\) (denoted \(W^\circ\)) if:
there is a \(\delta > 0\) such that \(B_d(w_0, \delta) \subseteq W\).

**Definition**
A point \(x_0 \in X\) is a **limit point** of \(W\) if there is:
a sequence of distinct elements, \(w_1, w_2, \ldots \in W\), that converges to \(x_0\).

**Definition**
A point \(x_0 \in X\) is in the **closure** of \(W\) (denoted \(\overline{W}\)) if:
for all \(\delta > 0\), there is a \(w_0 \in W\) such that \(d(x_0, w_0) < \delta\).

- The interior \(W^\circ\) is open (see definition)
- \(W\) is closed if and only if it contains all of its limit points
- Closure \(\overline{W}\) equals union of \(W\) and all its limit points (thus is closed)
Consider the standard metric space of real numbers $\mathbb{R}$

- Any open set can be written as countable disjoint union of open intervals
- But, what about closed sets?
- In higher dimensions, this fails for any connected set that is not a ball

The Boundary

- For $W \subseteq X$, the boundary $\partial W$ is the closure minus the interior $\overline{W} - W^\circ$
- Thus, the closure is the union of the set and its boundary
- In the figure, the boundary is the union of the dashed and solid lines
- Alternatively, a point $x \in X$ is on the boundary of $W$ if, for all $\delta > 0$, $B_d(x, \delta)$ contains a point in $W$ and a point not in $W$
Next Steps

To continue studying after this video –

- Try the suggested reading: Course Notes EF 2.1-2.1.2
- Or the optional reading: MMA 2.1
- Also, look at the problems in Assignment 3