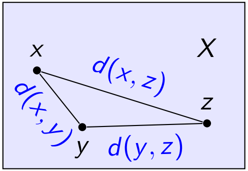
ECE 586: Vector Space Methods
Lecture 5 Flip Video: Metric Spaces and Topology

Henry D. Pfister
Duke University

2.1: Introduction

- What is topology and why do we study it?
  - Study of geometric properties preserved by continuous deformations
  - Why? Engineers approximate real things by mathematical objects
  - Q1: Can a matrix A be approximated closely by a lower rank matrix?
  - Q2: Can a function $f(x)$ be approximated well by a degree-2 polynomial?
  - In engineering, a topology is typically defined using a metric

- Metric Spaces
  - A metric space (X, d) is a set X along with a metric $d(x, y)$
  - $d(x, y)$ is called the distance between the points x and y
  - A metric on a set X is a function $d\colon X \times X \to \mathbb{R}$ such that:
    1. $d(x, y) \geq 0 \quad \forall x, y \in X$; with equality iff $x = y$ (non-negativity)
    2. $d(x, y) = d(y, x) \quad \forall x, y \in X$ (symmetry)
    3. $d(x, y) + d(y, z) \geq d(x, z) \quad \forall x, y, z \in X$ (triangle inequality)



2.1: Standard Examples of Metric Spaces

- Real numbers $X = \mathbb{R}$ with absolute distance $d(x, y) = |x - y|$
  - Foundation for real analysis: 1. and 2. are easy; 3. follows from

$$|x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z|$$

- $X = \mathbb{R}^n$ with Euclidean metric $d(\underline{x}, \underline{y}) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$
  - Properties 1. and 2. are easy; 3. is a bit harder and will be shown later

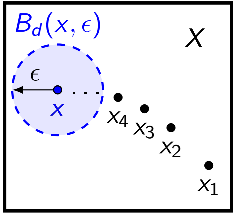
- Continuous functions $f : [a, b] \to \mathbb{R}$ with metric $d(f, g) = \max_{x \in [a,b]} |f(x) - g(x)|$
  - Properties 1. and 2. inherited from absolute distance; 3. follows from

$$\begin{aligned} \max_{x\in[a,b]} |f(x) - h(x)| &= \max_{x\in[a,b]} |f(x) - g(x) + g(x) - h(x)| \\ &\leq \max_{x\in[a,b]} [|f(x) - g(x)| + |g(x) - h(x)|] \\ &\leq \max_{x\in[a,b]} |f(x) - g(x)| + \max_{x\in[a,b]} |g(x) - h(x)| = d(f, g) + d(g, h) \end{aligned}$$

- "Set of points with distance less than ϵ from a point x"
  - This is called the open ball of radius ϵ centered at x and given by

$$B_d(x,\epsilon) \triangleq \{y \in X | d(x,y) < \epsilon\}$$

  - $P =$"For all $a \in B_d(x,\epsilon)$, there is $\delta > 0$ s.t. $B_d(a,\delta) \subseteq B_d(x,\epsilon)$"
- "Infinite list $x_1, x_2, x_3, \dots$ of points in X"
  - A sequence $x_i \in X$ for $i \in \mathbb{N}$ equivalent to $x_i = f(i)$ for $f : \mathbb{N} \to X$
  - Ex. For $X = \mathbb{R}$ and $d(x,y) = |x-y|$, let $x_n = \left(1 + \frac{1}{n}\right)^n$ for $n \in \mathbb{N}$
- "A sequence of points that approaches another point"
  - A sequence x_n converges to $x \in X$ (denoted $x_n \to x$) if, for any $\epsilon > 0$, there is natural number M such that $d(x, x_n) < \epsilon$ for all $n > M$



2.1: Convergence: Examples and Counterexamples

Definition

A sequence $x_1, x_2, \ldots$ in (X, d) is a Cauchy sequence if, for any $\epsilon > 0$, there is a natural number N (which may depend on ϵ) such that, for all $m, n > N$,

$$d(x_m, x_n) < \epsilon$$

- Theorem: Every convergent sequence is a Cauchy sequence
    - Proof will be given in the lecture
- Converse? No, there is a counterexample
    - Metric space (X, d) with rationals $X = \mathbb{Q}$ and $d(x, y) = |x - y|$
    - Sequence $x_1 = 2$ and $x_{n+1} = f(x_n) \triangleq \frac{1}{2}x_n + 1/x_n \in \mathbb{Q}$
    - One can show: x_n is a Cauchy sequence and $|x_n - \sqrt{2}| \to 0$
- But, according to the definitions, x_n does not converge!
    - Because convergence requires that the limit is in X but $\sqrt{2} \notin \mathbb{Q}$

2.1.1: Metric Topology

Definition

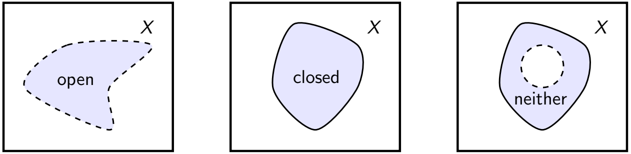
Let W be a subset of a metric space (X, d). The set W is called open if, for every $w_0 \in W$, there is an $\epsilon > 0$ such that $B_d(w_0, \epsilon) \subseteq W$.

Definition

A subset W of (X, d) is closed if its complement $W^c = X - W$ is open.

Theorem

1. *$\emptyset$ and X are open sets*
2. *any union of open sets is open*
3. *any finite intersection of open sets is open*



2.1.1: Interior, Limit points, and Closure

For a metric space (X, d) and subset $W \subseteq X$:

Definition

A point $w_0 \in W$ is in the interior of W (denoted W°) if:
there is a $\delta > 0$ such that $B_d(w_0, \delta) \subseteq W$.

Definition

A point $w \in X$ is a limit point of W if there is:
a sequence of distinct elements, $w_1, w_2, \ldots \in W$, that converges to w.

Definition

A point $x_0 \in X$ is in the closure of W (denoted $\overline{W}$) if:
for all $\delta > 0$, there is a $w_0 \in W$ such that $d(x_0, w_0) < \delta$.

- The interior W° is open (see definition)
- W is closed if and only if it contains all of its limit points
- Closure $\overline{W}$ equals union of W and all its limit points (thus is closed)

2.1.2: Continuity

Let $f\colon X \to Y$ be a function between metric spaces (X, d_X) and (Y, d_Y):

Definition

The function f is continuous at $x_0 \in X$ if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that, for all $x \in X$ satisfying $d_X(x_0, x) < \delta$, we find $d_Y\left(f(x_0), f(x)\right) < \epsilon$

Theorem

If f is continuous at x_0, then $f(x_n) \to f(x_0)$ for all sequences $x_1, x_2, \ldots \in X$ such that $x_n \to x_0$. Conversely, if $f(x_n) \to f(x_0)$ for all sequences $x_1, x_2, \ldots \in X$ such that $x_n \to x_0$, then f is continuous at x_0.

- f is called continuous if it is continuous at all $x_0 \in X$
- f is uniformly continuous if δ can be chosen independently of x_0

Definition

A function $f\colon X \to Y$ is called Lipschitz continuous on $A \subseteq X$ if there is a constant $L \in \mathbb{R}$ such that $d_Y(f(x), f(y)) \leq L\, d_X(x, y)$ for all $x, y \in A$.

Next Steps

- To continue studying after this video –
  - Try the suggested reading: Course Notes EF 2.1-2.1.2
  - Or the optional reading: MMA 2.1
  - Also, look at the problems in Assignment 3