

ECE 586: Vector Space Methods
Lecture 5 Flip Video: Metric Spaces and Topology

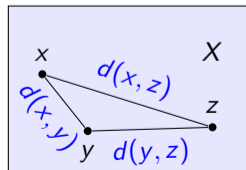
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2.1: Introduction

- What is topology and why do we study it?
 - Study of geometric properties preserved by continuous deformations
 - Why? Engineers approximate real things by mathematical objects
 - Q1: Can a matrix A be approximated closely by a lower rank matrix?
 - Q2: Can a function $f(x)$ be approximated well by a degree-2 polynomial?
 - In engineering, a topology is typically defined using a metric

- Metric Spaces

- A **metric space** (X, d) is a set X along with a metric $d(x, y)$
- $d(x, y)$ is called the **distance** between the **points** x and y
- A **metric** on a set X is a function $d: X \times X \rightarrow \mathbb{R}$ such that:
 1. $d(x, y) \geq 0 \quad \forall x, y \in X$; with equality iff $x = y$ (non-negativity)
 2. $d(x, y) = d(y, x) \quad \forall x, y \in X$ (symmetry)
 3. $d(x, y) + d(y, z) \geq d(x, z) \quad \forall x, y, z \in X$ (triangle inequality)



2.1: Standard Examples of Metric Spaces

- Real numbers $X = \mathbb{R}$ with **absolute distance** $d(x, y) = |x - y|$
 - Foundation for real analysis: 1. and 2. are easy; 3. follows from

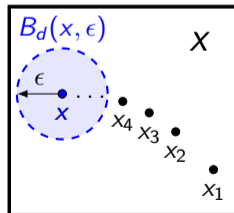
$$|x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z|$$

- $X = \mathbb{R}^n$ with **Euclidean metric** $d(\underline{x}, \underline{y}) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$
 - Properties 1. and 2. are easy; 3. is a bit harder and will be shown later
- Continuous functions $f: [a, b] \rightarrow \mathbb{R}$ with metric $d(f, g) = \max_{x \in [a, b]} |f(x) - g(x)|$
 - Properties 1. and 2. **inherited** from absolute distance; 3. follows from

$$\begin{aligned} \max_{x \in [a, b]} |f(x) - h(x)| &= \max_{x \in [a, b]} |f(x) - g(x) + g(x) - h(x)| \\ &\leq \max_{x \in [a, b]} [|f(x) - g(x)| + |g(x) - h(x)|] \\ &\leq \max_{x \in [a, b]} |f(x) - g(x)| + \max_{x \in [a, b]} |g(x) - h(x)| = d(f, g) + d(g, h) \end{aligned}$$

2.1: Important Concepts in Metric Spaces

- “Set of points with distance less than ϵ from a point x ”
 - This is called the **open ball** of radius ϵ centered at x and given by
$$B_d(x, \epsilon) \triangleq \{y \in X \mid d(x, y) < \epsilon\}$$
 - $P =$ “For all $a \in B_d(x, \epsilon)$, there is $\delta > 0$ s.t. $B_d(a, \delta) \subseteq B_d(x, \epsilon)$ ”
- “Infinite list x_1, x_2, x_3, \dots of points in X ”
 - A **sequence** $x_i \in X$ for $i \in \mathbb{N}$ equivalent to $x_i = f(i)$ for $f : \mathbb{N} \rightarrow X$
 - Ex. For $X = \mathbb{R}$ and $d(x, y) = |x - y|$, let $x_n = \left(1 + \frac{1}{n}\right)^n$ for $n \in \mathbb{N}$
- “A sequence of points that approaches another point”
 - A sequence x_n **converges** to $x \in X$ (denoted $x_n \rightarrow x$) if, for any $\epsilon > 0$, there is natural number M such that $d(x, x_n) < \epsilon$ for all $n > M$



2.1: Convergence: Examples and Counterexamples

Definition

A sequence x_1, x_2, \dots in (X, d) is a **Cauchy sequence** if, for any $\epsilon > 0$, there is a natural number N (which may depend on ϵ) such that, for all $m, n > N$,

$$d(x_m, x_n) < \epsilon$$

- Theorem: Every convergent sequence is a Cauchy sequence
 - Proof will be given in the lecture
- Converse? No, there is a counterexample
 - Metric space (X, d) with rationals $X = \mathbb{Q}$ and $d(x, y) = |x - y|$
 - Sequence $x_1 = 2$ and $x_{n+1} = f(x_n) \triangleq \frac{1}{2}x_n + 1/x_n \in \mathbb{Q}$
 - One can show: x_n is a Cauchy sequence and $|x_n - \sqrt{2}| \rightarrow 0$
- But, according to the definitions, x_n **does not converge!**
 - Because convergence requires that the limit is in X but $\sqrt{2} \notin \mathbb{Q}$

2.1.1: Metric Topology

Definition

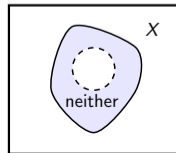
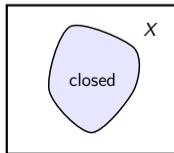
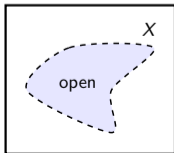
Let W be a subset of a metric space (X, d) . The set W is called **open** if, for every $w_0 \in W$, there is an $\epsilon > 0$ such that $B_d(w_0, \epsilon) \subseteq W$.

Definition

A subset W of (X, d) is **closed** if its complement $W^c = X - W$ is open.

Theorem

- 1 \emptyset and X are open sets
- 2 any union of open sets is open
- 3 any finite intersection of open sets is open



2.1.1: Interior, Limit points, and Closure

For a metric space (X, d) and subset $W \subseteq X$:

Definition

A point $w_0 \in W$ is in the **interior** of W (denoted W°) if:
there is a $\delta > 0$ such that $B_d(w_0, \delta) \subseteq W$.

Definition

A point $x_0 \in X$ is a **limit point** of W if there is:
a sequence of distinct elements, $w_1, w_2, \dots \in W$, that converges to x_0 .

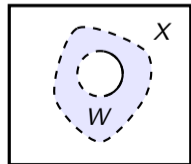
Definition

A point $x_0 \in X$ is in the **closure** of W (denoted \overline{W}) if:
for all $\delta > 0$, there is a $w_0 \in W$ such that $d(x_0, w_0) < \delta$.

- The interior W° is open (see definition)
- W is closed if and only if it contains all of its limit points
- Closure \overline{W} equals union of W and all its limit points (thus is closed)

A Few More Things

- Consider the standard metric space of real numbers \mathbb{R}
 - Any open set can be written as countable disjoint union of open intervals
 - But, what about closed sets?
 - In higher dimensions, this fails for any connected set that is not a ball
- The Boundary
 - For $W \subseteq X$, the **boundary** ∂W is the closure minus the interior $\overline{W} - W^\circ$
 - Thus, the closure is the union of the set and its boundary
 - In the figure, the boundary is the union of the dashed and solid lines
 - Alternatively, a point $x \in X$ is on the **boundary** of W if, for all $\delta > 0$, $B_d(x, \delta)$ contains a point in W and a point not in W



- To continue studying after this video –
 - Try the suggested reading: Course Notes EF 2.1-2.1.2
 - Or the optional reading: MMA 2.1
 - Also, look at the problems in Assignment 3