

ECE 586: Vector Space Methods
Lecture 6 Flip Video: Complete Metric Spaces

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2.1.3: Completeness

Definition

A metric space (X, d) is said to be **complete** if every Cauchy sequence in (X, d) converges to a limit $x \in X$.

Example

Consider the sequence $x_n \in \mathbb{Q}$ defined by $x_1 = 2$ and $x_{n+1} = \frac{1}{2}x_n + 1/x_n$. For this sequence, one can show that $|x_n - \sqrt{2}| \rightarrow 0$. Since $\sqrt{2}$ is not rational, however, this shows the standard **metric space of rationals is not complete**.

Key Point

The standard metric space of real numbers is a complete metric space. This can be proven by starting with Zermelo-Fraenkel (ZF) set theory and associating the real numbers with Cauchy sequences of rational numbers. The proof of this is not covered in this class but is available on the website.

2.1.3: Dense Subsets

Definition

A subset A of a metric space (X, d) is **dense** in X if every $x \in X$ is a limit point of the set A . This is equivalent to the closure \bar{A} being equal to X .

Example

The rational numbers are a dense subset of the real numbers because every real number is the limit of a sequence of rational numbers.

Definition

The **completion** of a metric space (X, d_X) consists of a complete metric space (Y, d_Y) and an isometry $\phi: X \rightarrow Y$ such that $\phi(X)$ is a dense subset of Y . Moreover, the completion is unique up to isometry.

Remark

One can always complete a metric space (X, d) by considering Cauchy sequences of elements in X .

2.1.3: A Space of Continuous Functions

Example

Let $X = C[-1, 1]$ be the space of continuous functions that map $[-1, 1]$ to \mathbb{R} and satisfy $\|f\|_2 < \infty$, where $\|f\|_2$ denotes the L^2 norm

$$\|f\|_2 \triangleq \left(\int_{-1}^1 |f(t)|^2 dt \right)^{1/2}$$

This set forms a metric space (X, d) when equipped with the distance

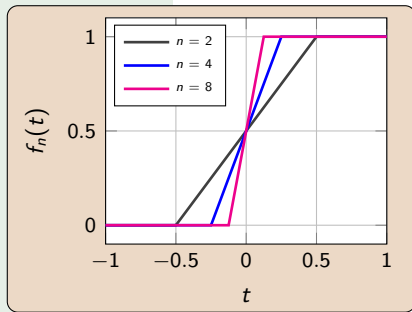
$$d(f, g) \triangleq \|f - g\|_2 = \left(\int_{-1}^1 |f(t) - g(t)|^2 dt \right)^{1/2}$$

Now, consider the sequence of functions $f_n(t)$ given by

$$f_n(t) \triangleq \begin{cases} 0 & t \in \left[-1, -\frac{1}{n}\right] \\ \frac{nt}{2} + \frac{1}{2} & t \in \left[-\frac{1}{n}, \frac{1}{n}\right] \\ 1 & t \in \left[\frac{1}{n}, 1\right] \end{cases}$$

One can show that **this is a Cauchy sequence in (X, d)**

But, **it does not converge** to a continuous function in $C[-1, 1]$



2.1.3: Contractions on Metric Spaces

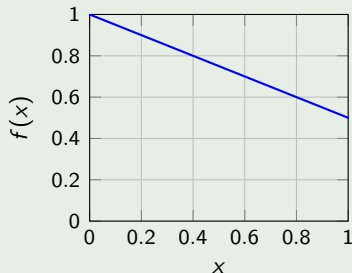
Definition

Let A be a subset of a metric space (X, d) and $f: X \rightarrow X$ be a function. Then, f is a **contraction** on A if $f(A) \subseteq A$ and there exists a constant $\gamma < 1$ such that $d(f(x), f(y)) \leq \gamma d(x, y)$ for all $x, y \in A$.

Example

Consider the metric space $X = [0, 1]$ with absolute distance. Define $f: X \rightarrow X$ with $f(x) = 1 - \frac{1}{2}x$ and observe that

$$d(f(x), f(y)) = |f(x) - f(y)| = \frac{1}{2}|x - y|$$



2.1.3: Contraction Mapping Theorem

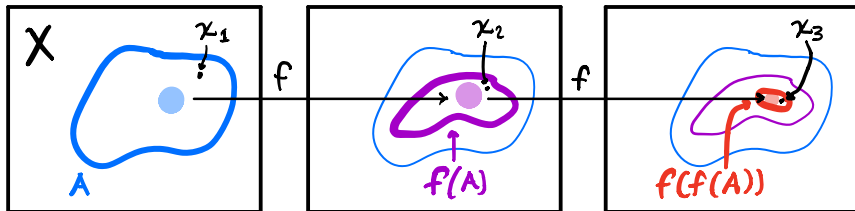
Theorem (Contraction Mapping Theorem)

Let (X, d) be a complete metric space and f be contraction on a closed subset $A \subseteq X$. Then,

- f has a unique fixed point $x^* \in A$ such that $f(x^*) = x^*$ and the sequence $x_{n+1} = f(x_n)$ converges to x^* from any initial $x_1 \in A$
- Also, x_n satisfies the error bounds (for contraction coefficient γ):

$$d(x^*, x_n) \leq \gamma^{n-1} d(x^*, x_1)$$

$$d(x^*, x_{n+1}) \leq d(x_n, x_{n+1})\gamma/(1 - \gamma)$$



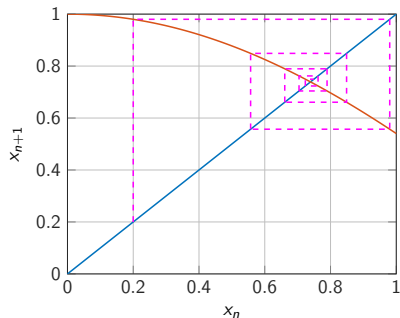
2.1.3: Applications of the Contraction Mapping Theorem

Several important results in applied mathematics have relatively simple proofs based on the contraction mapping theorem:

- **Picard's uniqueness theorem for differential equations**
 - Differential equation $y'(t) = f(t, y(t))$ for $t \in [a, b]$ with $y(a) = y_0$
 - Assume $f(t, y)$ is Lipschitz continuous in y for $t \in [a, b]$
 - Then, the solution $y(t)$ exists and is unique for $t \in [a, b]$
- **Implicit function theorem**
 - Let $f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ be continuously differentiable on an open set A
 - Let $g: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be defined implicitly by $f(x, g(x)) = 0$
 - For $x_0 \in A$, assume $f(x_0, y_0) = 0$ and y -Jacobian invertible at (x_0, y_0)
 - Then, $g(x)$ exists and is unique in some neighborhood of x_0
- **Dynamic Programming for a Markov Decision Process (MDP)**
 - State-action (s, a) defines probability $p(s'|s, a)$ and reward $R(s, a)$
 - Finite state + discounted reward \Rightarrow unique stationary optimal policy

2.1.3: Concrete Example of the Contraction Mapping Theorem

Starting from $x_1 = 0.2$, define the sequence $x_{n+1} = \cos(x_n)$ and plot the x - y pairs (x_n, x_{n+1}) . Each point is connected to the “ $y = x$ ” line to emphasize the sequential path.



- Let $X = [0, 1]$ and define $f: X \rightarrow X$ via $f(x) = \cos(x)$
- $\cos([0, 1]) = [\cos(1), 1]$ because $\cos(x)$ decreasing on $[0, \pi]$
- Mean value theorem: $f(y) - f(x) = f'(t)(y - x)$ for some $t \in [x, y]$
- $f'(t) = -\sin(t)$ and $\sin([0, 1]) = [0, \sin(1)]$ with $\sin(1) \approx 0.84$
- $|\cos(y) - \cos(x)| \leq 0.85 |y - x| \Rightarrow f(x)$ is a contraction on $[0, 1]$
- $x_{n+1} = \cos(x_n)$ converges to unique fixed point $x^* = \cos(x^*) \approx 0.739$

- To continue studying after this video –
 - Try the suggested reading: Course Notes EF 2.1.3
 - Or the optional reading: MMA 2.1
 - Also, look at the problems in Assignment 3