ECE 586: Vector Space Methods
Lecture 6 Flip Video: Real Numbers and Continuity

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2.1.4: Properties of Real Numbers

- Suppose we include the natural boundary values for the real numbers
  - This gives the extended real numbers $\overline{\mathbb{R}} \triangleq \mathbb{R} \cup \{\infty, -\infty\}$
  - $\overline{\mathbb{R}}$ forms a metric space with metric $d_{\overline{\mathbb{R}}}(x, y) \triangleq \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right|$
  - "$x_n \to \infty$" is equivalent to "$\forall M > 0,\ \exists N \in \mathbb{N},\ \forall n > N,\ x_n > M$"

Definition

The supremum (or least upper bound) of $X \subseteq \mathbb{R}$ is denoted $\sup X$ and equals the smallest extended real number $M \in \overline{\mathbb{R}}$ such that $x \leq M$ for all $x \in X$.

Lemma (supremum sequence)

Let X be a metric space and $f : X \to \mathbb{R}$ be a function mapping X to the real numbers. Let $M = \sup f(A)$ for some non-empty $A \subseteq X$. Then, there exists a sequence $x_1, x_2, \ldots \in A$ such that $\lim_n f(x_n) = M$.

- Sketch proof on the board

2.1.4: More Properties of Real Numbers

Definition

The maximum of $X \subseteq \mathbb{R}$, denoted $\max X$, is the largest value contained in the set. It equals the supremum if $\sup X \in X$ and it is undefined otherwise.

Example

$X = [1, 2) \subset \mathbb{R}$ has $\sup X = 2$ and $\max X$ undefined.

For $f(x) = \frac{1}{2-x}$, $f(X) = [1, \infty)$ and $\sup f(X) = \infty$.

- infimum: $\inf X = -(\sup -X)$, where $-X = \{x \in \mathbb{R} \,|\, -x \in X\}$
- minimum: $\min X = -(\max -X)$, if it exists
- supremum and infimum always well-defined but may equal $\pm\infty$

Theorem

A bounded non-decreasing sequence in $\mathbb{R}$ converges to its supremum.

Sketch proof and application to sums on the board

Let $f : X \to Y$ be a function between metric spaces (X, d_X) and (Y, d_Y):

Definition

The function f is continuous at $x_0 \in X$ if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that, for all $x \in X$ satisfying $d_X(x_0, x) < \delta$, we find $d_Y(f(x_0), f(x)) < \epsilon$

Theorem

If f is continuous at x_0, then $f(x_n) \to f(x_0)$ for all sequences $x_1, x_2, \ldots \in X$ such that $x_n \to x_0$. Conversely, if $f(x_n) \to f(x_0)$ for all sequences $x_1, x_2, \ldots \in X$ such that $x_n \to x_0$, then f is continuous at x_0.

- f is called continuous if it is continuous at all $x_0 \in X$
- f is uniformly continuous if δ can be chosen independently of x_0

Continuous vs. Uniformly Continuous

Let $X = (0, 1]$ and $Y = [1, \infty)$ be subsets of the standard metric space $\mathbb{R}$

- Let $f : X \to Y$ be defined by $f(x) = \frac{1}{x}$
  - Is this function continuous? What could go wrong?
  - For all $x_0 \in X$ and $\epsilon > 0$, we can choose $\delta = \frac{\epsilon x_0^2}{1+\epsilon x_0}$ and observe that

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x_0 - x}{x\,x_0}\right| < \frac{\frac{\epsilon x_0^2}{1+\epsilon x_0}}{x_0\left(x_0 - \frac{\epsilon x_0^2}{1+\epsilon x_0}\right)} = \frac{\epsilon}{(1+\epsilon x_0)(1 - \frac{\epsilon x_0}{1+\epsilon x_0})} = \epsilon$$

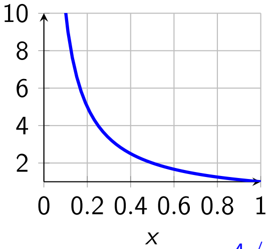
  - It is uniformly continuous? Where could something go wrong?
  - Negating the definition of uniformly continuous gives

$$\exists \epsilon > 0, \forall \delta > 0, \exists x_0 \in X, \exists x \in X \text{ s.t. } |x - x_0| < \delta, |f(x) - f(x_0)| \geq \epsilon$$

If $\epsilon = 1$, $x_0 = \min\left\{\frac{1}{2}, \frac{\delta}{2}\right\}$, $x = 2x_0$, then $|x - x_0| = x_0 \leq \frac{\delta}{2}$ and

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x_0 - x}{x\,x_0}\right| = \frac{x_0}{x\,x_0} = \frac{1}{x} = \max\left\{1, \frac{1}{\delta}\right\} \geq 1$$

$f(x) = \frac{1}{x}$



Uniformly Continuous vs. Lipschitz Continuous

Definition

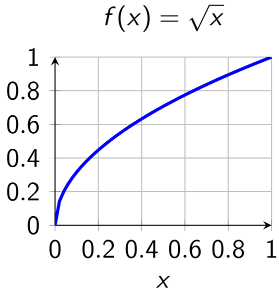
A function $f\colon X \to Y$ is called Lipschitz continuous on $A \subseteq X$ if there is a constant $L \in \mathbb{R}$ such that $d_Y(f(x), f(y)) \leq L\, d_X(x, y)$ for all $x, y \in A$.

Let $X = [0, 1]$ and $Y = [0, 1]$ be subsets of the standard metric space $\mathbb{R}$

- Let $f : X \to Y$ be defined by $f(x) = \sqrt{x}$
  - Is this function Lipschitz continuous? What could go wrong?
  - Lower bound the Lipschitz constant via $x \mapsto z^2$,$y \mapsto z^2 + z$:

$$L \geq \sup_{\substack{x,y\in X\\ x\neq y}} \frac{|\sqrt{y}-\sqrt{x}|}{|y-x|} \geq \sup_{z\in(0,\frac{1}{2}]} \frac{\sqrt{z^2+z}-\sqrt{z^2}}{(z^2+z)-z^2} \geq \sup_{z\in(0,\frac{1}{2}]} \frac{1}{\sqrt{z}} - 1$$

  - Exercise: Is this function uniformly continuous? (Try $\delta = \epsilon^2$)



2.1.5: Sequences of Functions

Let (X, d_X) and (Y, d_Y) be metric spaces and $f_n : X \to Y$ for $n \in \mathbb{N}$ be a sequence of functions mapping X to Y.

Definition

The sequence f_n converges pointwise to $f : X \to Y$ if, for all $x \in X$,

$$\lim_{n \to \infty} f_n(x) = f(x)$$

Definition

The sequence f_n converges uniformly to $f : X \to Y$ if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, \forall x \in X, d_Y(f_n(x), f(x)) < \epsilon.$$

Theorem

If each f_n is continuous and the sequence f_n converges uniformly to $f : X \to Y$, then the limit function f is continuous.

Next Steps

- To continue studying after this video –
  - Try the suggested reading: Course Notes EF 2.1.3 - 2.1.5
  - Or the optional reading: MMA 2.1
  - Also, look at the problems in Assignment 3