ECE 586: Vector Space Methods
Lecture 7 Flip Video: Compactness

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2.1.4: Compactness

Definition

A metric space (X, d) is totally bounded if, for any $\epsilon > 0$, there exists a finite set of radius-ϵ balls that cover X (i.e., $\cup_{x \in S} B_d(x, \epsilon) = X$).

Definition

A metric space is compact if it is complete and totally bounded.

- Examples
  - The closed real interval $[0, 1] \subset \mathbb{R}$ is compact
  - A subset of Euclidean $\mathbb{R}^n$ is compact iff it is closed and bounded
  - But, the standard metric space of real numbers is not compact because it is not totally bounded.

Theorem

A closed subset A of a compact space X is itself a compact space.

2.1.4: Compactness and Sequences

Definition

Let $x_1, x_2, \ldots \in X$ be a sequence and $n_1, n_2, \ldots \in \mathbb{N}$ be a strictly increasing sequence. Then, $x_{n_1}, x_{n_2}, \ldots$ is called subsequence.

Theorem

A sequence in a compact metric space has a subsequence that converges.

Example

For the compact metric space $X = [-2, 2] \subset \mathbb{R}$ with absolute distance, let $x_n = (-1)^n + \frac{1}{n}$. Then, subsequence $x_2, x_4, x_6, \ldots$ converges to 1.

- Sketch proof on whiteboard in pictures

2.1.4: Properties of Real Numbers

- Let us include the natural boundary values for the real numbers
  - Extended Real Numbers: $\overline{\mathbb{R}} \triangleq \mathbb{R} \cup \{\infty, -\infty\}$
  - This is a compact metric space with metric $d_{\overline{\mathbb{R}}}(x, y) \triangleq \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right|$
  - "$x_n \to \infty$" equivalent to "$\forall M > 0, \exists N \in \mathbb{N}, \forall n > N, x_n > M$"

Definition

The supremum (or least upper bound) of $X \subseteq \mathbb{R}$ is denoted $\sup X$ and equals the smallest extended real number $M \in \overline{\mathbb{R}}$ such that $x \leq M$ for all $x \in X$.

Lemma (supremum sequence)

Let X be a metric space and $f : X \to \mathbb{R}$ be a function mapping X to the real numbers. Let $M = \sup f(A)$ for some non-empty $A \subseteq X$. Then, there exists a sequence $x_1, x_2, \ldots \in A$ such that $\lim_n f(x_n) = M$.

- Sketch proof on whiteboard

2.1.4: More Properties of Real Numbers

Definition

The maximum of $X \subseteq \mathbb{R}$, denoted $\max X$, is the largest value contained in the set. It equals the supremum if $\sup X \in X$ and it is undefined otherwise.

Example

$X = [1, 2) \subset \mathbb{R}$ has $\sup X = 2$ and $\max X$ undefined.

For $f(x) = \frac{1}{2-x}$, $f(X) = [1, \infty)$ and $\sup f(X) = \infty$.

- infimum: $\inf X = -(\sup -X)$, where $-X = \{x \in \mathbb{R} \mid -x \in X\}$
- minimum: $\min X = -(\max -X)$, if it exists
- supremum and infimum always well-defined but may equal $\pm\infty$

Theorem

A bounded non-decreasing sequence of real numbers converges to its supremum.

2.1.5: Sequences of Functions

Let (X, d_X) and (Y, d_Y) be metric spaces and $f_n \colon X \to Y$ for $n \in \mathbb{N}$ be a sequence of functions mapping X to Y.

Definition

The sequence f_n converges pointwise to $f \colon X \to Y$ if, for all $x \in X$,

$$\lim_{n \to \infty} f_n(x) = f(x)$$

Definition

The sequence f_n converges uniformly to $f \colon X \to Y$ if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, \forall x \in X, \; d_Y(f_n(x), f(x)) < \epsilon.$$

Theorem

If each f_n is continuous and f_n converges uniformly to $f \colon X \to Y$, then f is continuous.

Theorem

Let X be a metric space and $f: X \to \mathbb{R}$ be a continuous function from X to $\mathbb{R}$. If A is a compact subset of X, then there exists $x \in A$ such that $f(x) = \sup f(A)$ (i.e., f achieves a maximum on A).

Theorem

Let (X, d) be a compact metric space and $C(X)$ be the set of continuous functions mapping X to $\mathbb{R}$. This set of functions, with the metric

$$d_\infty(f, g) \triangleq \sup_{x \in X} |f(x) - g(x)| = \max_{x \in X} |f(x) - g(x)|,$$

defines a complete metric space.

Note: For $f, g \in C(X)$, d_∞ metrizes uniform convergence because

$$\text{“}\max_{x \in X} |f(x) - g(x)| < \epsilon\text{”} \Leftrightarrow \text{“}\forall x \in X, |f(x) - g(x)| < \epsilon\text{”}.$$

Next Steps

- To continue studying after this video –
  - Try the suggested reading: Course Notes EF 2.1.4 - 2.1.5
  - Or the optional reading: MMA 2.1
  - Also, look at the problems in Assignment 3