2.1.4: Compactness

**Definition**

A metric space \((X, d)\) is **totally bounded** if, for any \(\epsilon > 0\), there exists a finite set of radius-\(\epsilon\) balls that cover \(X\) (i.e., \(\bigcup_{x \in S} B_d(x, \epsilon) = X\)).

**Definition**

A metric space is **compact** if it is complete and totally bounded.

- **Examples**
  - The closed real interval \([0, 1] \subset \mathbb{R}\) is compact
  - A subset of Euclidean \(\mathbb{R}^n\) is compact iff it is closed and bounded
  - But, the standard metric space of real numbers is not compact because it is not totally bounded.

**Theorem**

*A closed subset \(A\) of a compact space \(X\) is itself a compact space.*
2.1.4: Compactness and Sequences

**Definition**
Let $x_1, x_2, \ldots \in X$ be a sequence and $n_1, n_2, \ldots \in \mathbb{N}$ be a strictly increasing sequence. Then, $x_{n_1}, x_{n_2}, \ldots$ is called subsequence.

**Theorem**
A sequence in a compact metric space has a subsequence that converges.

**Example**
For the compact metric space $X = [-2, 2] \subset \mathbb{R}$ with absolute distance, let $x_n = (-1)^n + \frac{1}{n}$. Then, subsequence $x_2, x_4, x_6, \ldots$ converges to 1.

- Sketch proof on whiteboard in pictures
2.1.4: Properties of Real Numbers

- Let us include the natural boundary values for the real numbers
  - Extended Real Numbers: $\mathbb{R} \triangleq \mathbb{R} \cup \{\infty, -\infty\}$
  - This is a compact metric space with metric $d_{\mathbb{R}}(x, y) \triangleq \frac{|x|}{1+|x|} - \frac{|y|}{1+|y|}$
  - "$x_n \to \infty$" equivalent to "$\forall M > 0, \exists N \in \mathbb{N}, \forall n > N, x_n > M$"

**Definition**

The **supremum** (or least upper bound) of $X \subseteq \mathbb{R}$ is denoted $\text{sup } X$ and equals the smallest extended real number $M \in \mathbb{R}$ such that $x \leq M$ for all $x \in X$.

**Lemma (supremum sequence)**

Let $X$ be a metric space and $f: X \to \mathbb{R}$ be a function mapping $X$ to the real numbers. Let $M = \text{sup } f(A)$ for some non-empty $A \subseteq X$. Then, there exists a sequence $x_1, x_2, \ldots \in A$ such that $\lim_n f(x_n) = M$.

- Sketch proof on whiteboard
2.1.4: More Properties of Real Numbers

**Definition**
The maximum of $X \subseteq \mathbb{R}$, denoted $\max X$, is the largest value contained in the set. It equals the supremum if $\sup X \in X$ and it is undefined otherwise.

**Example**
$X = [1, 2) \subset \mathbb{R}$ has $\sup X = 2$ and $\max X$ undefined.
For $f(x) = \frac{1}{2-x}$, $f(X) = [1, \infty)$ and $\sup f(X) = \infty$.

- **infimum**: $\inf X = - (\sup -X)$, where $-X = \{ x \in \mathbb{R} \mid -x \in X \}$
- **minimum**: $\min X = - (\max -X)$, if it exists
- supremum and infimum always well-defined but may equal $\pm \infty$

**Theorem**
A bounded non-decreasing sequence of real numbers converges to its supremum.
Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces and \(f_n : X \to Y\) for \(n \in \mathbb{N}\) be a sequence of functions mapping \(X\) to \(Y\).

**Definition**

The sequence \(f_n\) **converges pointwise** to \(f : X \to Y\) if, for all \(x \in X\),

\[
\lim_{n \to \infty} f_n(x) = f(x)
\]

**Definition**

The sequence \(f_n\) **converges uniformly** to \(f : X \to Y\) if

\[
\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N, \forall x \in X, \; d_Y(f_n(x), f(x)) < \varepsilon.
\]

**Theorem**

*If each \(f_n\) is continuous and \(f_n\) converges uniformly to \(f : X \to Y\), then \(f\) is continuous.*
2.1.5: Two Important Results

**Theorem**

Let $X$ be a metric space and $f : X \rightarrow \mathbb{R}$ be a continuous function from $X$ to $\mathbb{R}$. If $A$ is a compact subset of $X$, then there exists $x \in A$ such that $f(x) = \sup f(A)$ (i.e., $f$ achieves a maximum on $A$).

**Theorem**

Let $(X, d)$ be a compact metric space and $C(X)$ be the set of continuous functions mapping $X$ to $\mathbb{R}$. This set of functions, with the metric

$$d_\infty(f, g) \triangleq \sup_{x \in X} |f(x) - g(x)| = \max_{x \in X} |f(x) - g(x)|,$$

defines a complete metric space.

Note: For $f, g \in C(X)$, $d_\infty$ metrizes uniform convergence because

"$\max_{x \in X} |f(x) - g(x)| < \epsilon$" $\iff$ "$\forall x \in X, |f(x) - g(x)| < \epsilon$".
Next Steps

To continue studying after this video –

- Try the suggested reading: Course Notes EF 2.1.4 - 2.1.5
- Or the optional reading: MMA 2.1
- Also, look at the problems in Assignment 3