ECE 586: Vector Space Methods
Lecture 8 Flip Video: Compactness

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Definition

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2.1.4: Compactness

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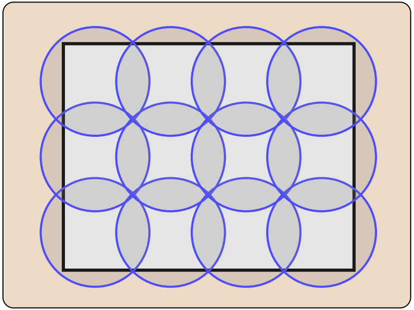

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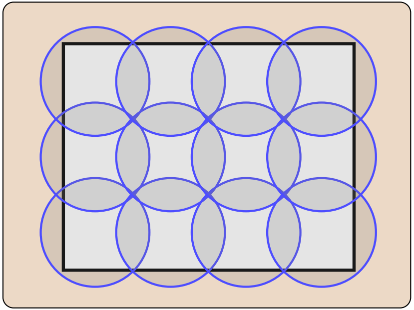

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- Examples
  - The closed real interval $[0, 1] \subset \mathbb{R}$ is compact
  - A subset of Euclidean $\mathbb{R}^n$ is compact iff it is closed and bounded
  - But, the standard metric space of real numbers is not compact because it is not totally bounded.

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Theorem

A closed subset A of a compact space X is itself a compact space.

2.1.4: Compactness and Sequences

Definition

Let $x_1, x_2, \ldots \in X$ be a sequence and $n_1, n_2, \ldots \in \mathbb{N}$ be a strictly increasing sequence (i.e., $n_{i+1} > n_i$). Then, $x_{n_k} = (x_{n_1}, x_{n_2}, \ldots)$ is a subsequence of x_n.

Theorem (Bolzano–Weierstrass)

A sequence in a compact metric space has a subsequence that converges.

Example

For the compact metric space $X = [-2, 2] \subset \mathbb{R}$ with absolute distance, let $x_n = (-1)^n + \frac{1}{n}$. Then, the subsequence $x_2, x_4, x_6, \ldots$ converges to 1.

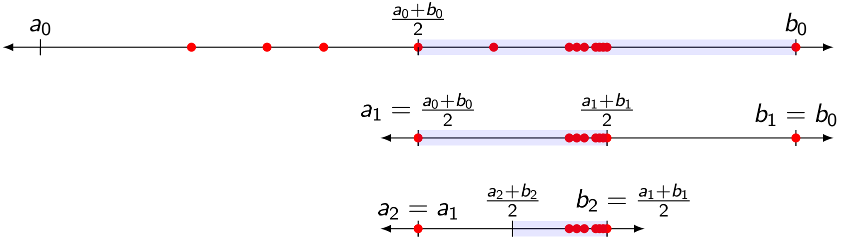
- Does the sequence x_n have any other limit points?

2.1.4: Bolzano–Weierstrass for a Closed Real Interval

$I_0 = X = [a_0, b_0]$

$I_1 = [a_1, b_1]$

$I_2 = [a_2, b_2]$



- For $X = [a_0, b_0] \subseteq \mathbb{R}$ with $d(x, y) = |x - y|$, let $x_n^{(0)} \in X$ be the red dots
- Observe $|\{n \in \mathbb{N} \mid x_n^{(0)} \in I_0\}| = \infty$ and split interval I_0 in half at $\frac{a_0+b_0}{2}$
- Of the two halves, at least one must contain infinitely many elements of $x_n^{(0)}$
- Call it I_1 and let $x_n^{(1)}$ be the subseq of $x_n^{(0)}$ in I_1. Then, $|\{n \in \mathbb{N} \mid x_n^{(1)} \in I_1\}| = \infty$
- Of the two halves of I_1, one must contain infinitely many elements of $x_n^{(1)}$
- Continuing by induction, the length of I_k is $|b_k - a_k| = 2^{-k}|b_0 - a_0|$
- Using this, one can show $x_k^{(k)}$ is a subsequence of x_n that is Cauchy

2.1.5: Two Important Consequences

Theorem

Let X be a metric space and $f : X \to \mathbb{R}$ be a continuous function from X to $\mathbb{R}$. If A is a compact subset of X, then there exists $x \in A$ such that $f(x) = \sup f(A)$ (i.e., f achieves a maximum on A).

Theorem

Let (X, d) be a compact metric space and $C(X)$ be the set of continuous functions mapping X to $\mathbb{R}$. This set of functions, with the metric

$$d_\infty(f, g) \triangleq \sup_{x \in X} |f(x) - g(x)| = \max_{x \in X} |f(x) - g(x)|,$$

defines a complete metric space.

Note: For $f, g \in C(X)$, d_∞ metrizes uniform convergence because

$$\text{“} \max_{x \in X} |f(x) - g(x)| < \epsilon \text{”} \Leftrightarrow \text{“} \forall x \in X, |f(x) - g(x)| < \epsilon \text{”}.$$

Next Steps

- To continue studying after this video –
  - Try the suggested reading: Course Notes EF 2.1.4 - 2.1.5
  - Or the optional reading: MMA 2.1
  - Also, look at the problems in Assignment 3