ECE 586: Vector Space Methods
Lecture 9 Flip Video: Fields and Matrices

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3.1: Linear Algebra and Abstraction

- Linear algebra is the foundation of engineering mathematics
  - This chapter will review and extend your knowledge of linear algebra
  - Abstraction allows the application of similar ideas to many different areas
  - Your prior experience with real vector spaces makes this possible
- Abstraction
  - Metric spaces provide abstract notions of distance and closeness
  - Fields provide abstract notions for arithmetic of scalars $(+, -, \times, /)$
  - Vector spaces abstract the additive properties of real vectors

3.1: Fields

Consider a set F with binary operations: "addition" and "multiplication"

- For every pair of elements $s, t \in F$, we require $(s + t) \in F$
- For every pair of elements $s, t \in F$, we require $st \in F$

F forms a field if the two operations satisfy:

1. there is a unique element $0 \in F$ such that $s + 0 = s \quad \forall s \in F$
2. addition is commutative: $s + t = t + s \quad \forall s, t \in F$
3. addition is associative: $r + (s + t) = (r + s) + t \quad \forall r, s, t \in F$
4. for $s \in F$ there is a unique element $(-s) \in F$ such that $s + (-s) = 0$
5. multiplication is commutative: $st = ts \quad \forall s, t \in F$
6. multiplication is associative: $r(st) = (rs)t \quad \forall r, s, t \in F$
7. there is a unique non-zero element $1 \in F$ such that $s1 = s \quad \forall s \in F$
8. for $s \in F \setminus \{0\}$ there is a unique element $s^{-1} \in F$ such that $ss^{-1} = 1$
9. mult. distributes over addition: $r(s + t) = rs + rt \quad \forall r, s, t \in F$.

3.1: Examples of Arithmetic Fields

Example

The rational numbers with standard addition and multiplication form a field.

Example

The real numbers with standard addition and multiplication form a field.

Example

The complex numbers with standard addition and multiplication form a field.

Example

The set of integers with standard addition and multiplication is not a field.

Example

For prime p, set $\{0, 1, \ldots, p-1\}$ with add. and mult. modulo p is a field.

Consider finding n scalars $x_1, \ldots, x_n \in F$ that satisfy these conditions

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & y_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & y_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & y_m \end{array}$$

They define a system of m linear equations in n unknowns $A\underline{x} = \underline{y}$, where $\underline{x} = (x_1, \ldots, x_n)^T$, $\underline{y} = (y_1, \ldots, y_m)^T$, and A is the matrix given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Definition

Let A be an $m \times n$ matrix over F (i.e., $A \in F^{m \times n}$) and B be an $n \times p$ matrix over F. The matrix product AB is the $m \times p$ matrix C whose i, j entry is

$$c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}.$$

Example

When j is fixed, one can eliminate i by grouping the elements of C and A into column vectors $\underline{c}_1, \ldots, \underline{c}_p$ and $\underline{a}_1, \ldots, \underline{a}_n$. In this case, we see that the j-th column of C is a linear combination of the columns of A,

$$\underline{c}_j = \sum_{r=1}^{n} \underline{a}_r b_{rj},$$

Likewise, grouping C and B into row vectors $\underline{c}_1, \ldots, \underline{c}_m$ and $\underline{b}_1, \ldots, \underline{b}_n$ gives

$$\underline{c}_i = \sum_{r=1}^{n} a_{ir} \underline{b}_r.$$

3.2: Echelon Forms

Definition

A matrix $A \in F^{m\times n}$ is in row echelon form if:

1. Any all-zero row is below all rows with non-zero entries, and
2. For all rows with non-zero entries, the leading coefficient (i.e., the first non-zero element from the left) is strictly to the right of the leading coefficient of the row above it.

Thus, the entries below the leading coefficient in a column are zero.
A is in column echelon form if A^T is in row echelon form.

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition

A matrix $B \in F^{m\times n}$ is in reduced row echelon form (RREF) if it is:

1. in row echelon form with every leading coefficient equal to 1, and
2. every leading coefficient is the only non-zero element in its column.

B is reduced column echelon form if B^T is in reduced row echelon form.

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3.2: Elementary Row Operations

Definition

An elementary row operation on an $m \times n$ matrix consists of

1. multiplying a row by a non-zero scalar,
2. swapping two rows, or
3. adding a scalar multiple of one row to another row.

An elementary column operation is the same but applied to the columns.

Definition

A square matrix $A \in F^{m \times m}$ is invertible if there is a square matrix $B \in F^{m \times m}$ such that $AB = BA = I$, where I is the $m \times m$ identity. In that case, $A^{-1} = B$.

- Elementary row operations on $A \in F^{m \times n}$
  - Are realized, for an invertible matrix $E \in F^{m \times m}$, by matrix mult as EA
  - Can be used in a sequence to induce RREF (e.g., Gaussian elimination)

3.2: Elementary Row Operations

Lemma

For any $m \times n$ matrix A over F, there is an invertible $m \times m$ matrix P over F such that $R = PA$ is in reduced row echelon form.

Sketch of Proof.

- For $A \in F^{m \times n}$, define the augmented matrix $A' = [A \;\; I]$
- Use elementary row operations to transform A' into RREF as R'
- Elementary row operations imply that $R' = PA'$ for some invertible P
- Then, $R' = [R \;\; P]$ where $R = PA$ is in RREF □

3.2: Foundation of Dimension

Lemma

Let A be an $m \times n$ matrix over F with $m < n$. Then, there exists a length-n column vector $\underline{x} \neq \underline{0}$ (over F) such that $A\underline{x} = \underline{0}$.

Proof.

- Use row reduction to compute the reduced row echelon form $R = PA$ of A, where P is invertible by previous lemma.
- Observe that the columns of R containing leading elements can be combined in a linear combination to cancel any other column of R.
- Thus, one can construct a vector $\underline{x}$ satisfying $R\underline{x} = \underline{0}$ and observe that $A\underline{x} = P^{-1}R\underline{x} = \underline{0}$. □

A concrete example of this process will be given in class.

Next Steps

- To continue studying after this video –
  - Try the required reading: Course Notes EF 3.1 - 3.3
  - Or the recommended reading: LADR Ch. 1, Ch. 2, Ch. 3ABC
  - Also, look at the problems in Assignment 4