Graphical Representation of Error-Correcting Codes

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Introduction to Error-Correcting Codes
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Trellis Representation of Codes

Definition (Trellis Diagram)

A trellis diagram with n sections is a labeled directed graph with the following properties:

- vertices are grouped into $n + 1$ layers and labeled by their layer number $\ell \in \{0, 1, \ldots, n\}$;
- edges are grouped into sections connecting adjacent layers and labeled by the alphabet $\mathcal{X}$.

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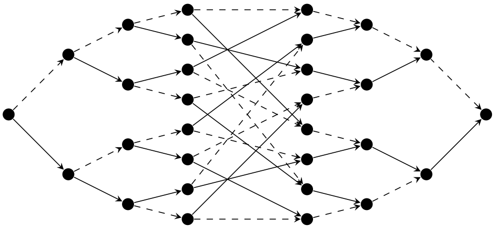
A trellis diagram represents a length-n code $\mathcal{C} \subseteq \mathcal{X}^n$ by providing a bjiective mapping between codewords and directed paths from layer 0 to layer n. In particular, each codeword $\underline{x} \in \mathcal{X}^n$ equals the sequence of edge labels along a path from layer 0 to layer n.



Trellis diagram for (7,4) Hamming code where dashed edges represent 1's

Trellis Construction

- Canonical trellis for an (n, k) binary linear code from parity-check matrix $H = [\underline{h}_1, \ldots, \underline{h}_n]$
  - Vertices are labeled by layer (i.e., $\ell \in \{0, 1, \ldots, n\}$) and partial syndrome $\underline{s} \in \{0, 1\}^{n-k}$
  - For codeword $\underline{x} \in \mathcal{C}$, the directed path contains the layer-ℓ vertex with partial syndrome

$$\underline{s}_\ell(\underline{x}) = \sum_{i=1}^{\ell} x_i \underline{h}_i$$

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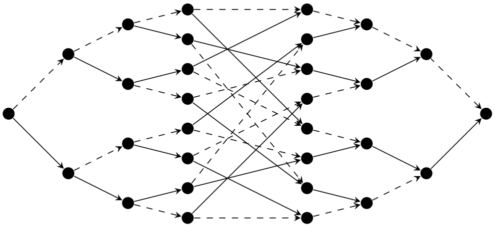
Consider the $(7, 4)$ binary Hamming code defined by the parity-check matrix

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$



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Trellis Decoding

- Setup
  - A uniform random codeword $\underline{X} \in \{0,1\}^n$ is transmitted
  - A realization $\underline{y}$ of the random vector $\mathcal{Y} \in \mathcal{Y}^n$ is received through a memoryless channel
  - The optimal decoder chooses the codeword $\hat{\underline{x}}$ that maximizes the likelihood

$$\mathbb{P}(\underline{Y} = \underline{y} | \underline{X} = \underline{x}) = \prod_{i=1}^{n} \mathbb{P}(Y_i = y_i | X_i = x_i)$$

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- Optimal decoding
  - Define weighted trellis where an edge in the i-th section with bit label x has weight

$$w_\ell(x) \triangleq -\ln \mathbb{P}\big(Y_i = y_i | X_i = x\big).$$

  - Then, the most likely codeword minimizes the sum of the weights along its path and, thus, we define the weight of the directed path for $\underline{x} = (x_1, x_2, \ldots, x_n)$ to be additive

$$w(\underline{x}) \triangleq \sum_{i=1}^{n} w_i(x_i)$$

  - Efficient implementation via dynamic programming called the Viterbi algorithm

Tanner Graph of a Code

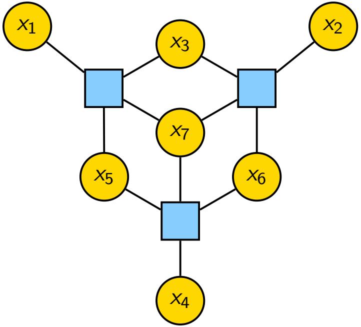
Definition (Tanner Graph)

The Tanner graph G of an $m \times n$ parity-check matrix H is a bipartite graph with one vertex for each code symbol and one vertex for each parity check. For each non-zero entry in H, the graph contains an edge that connects the variable node associated with the column to the check node associated with the row.

Mathematically, we have $G = (V \cup C, E)$ with

$$V = \{1, 2, \ldots n\} \qquad C = \{1, 2, \ldots, m\}$$

$$E = \{(i, j) \in C \times V \mid H_{i,j} \neq 0\}.$$



$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Graph Peeling on the Tanner Graph

For a codeword observed through the binary erasure channel (BEC), the peeling decoder iteratively removes known bits from the graph as follows:

1. Initialize the variables $x_1, \ldots, x_n$ to ? and the variables $y_1, \ldots, y_m$ to zero.
2. For each correct code symbol, let $j \in V$ be its index and set variable x_j to the known value.
3. If there is a degree-1 check node, let $i \in C$ be its index, $j \in V$ be the index of the adjacent variable node, and set $x_j = H_{ij}^{-1} y_i$.
4. If graph contains a variable node with known value (i.e., $x_j \neq ?$), let $j \in V$ be its index and
    1. for all i such that $(i, j) \in E$, update $y_i = y_i - H_{ij} x_j$
    2. remove bit j and all adjacent edges (i.e., $V \leftarrow V \backslash j$, $E \leftarrow E \backslash \{(i, j') \in E \,|\, j = j'\}$)
    3. Goto step 3
5. When the algorithm reaches this point, either decoding is successful and $x_j \neq ?$ for all $j \in V$ or the decoder is stuck in a stopping set where there are no degree-1 check nodes and the graph contains only variable nodes whose values are unknown.