# ECEN 314: Signals and Systems

## Lecture Notes 10: Discrete-Time Fourier Series

**Reading:**

- Current: SSOW 3.6-3.7
- Next: SSOW 3.9

# 1 Periodic DT signals

A DT signal $x[n]$ is said to be *periodic* if there exists a positive *integer* $N$ such that

$$x[n+N] = x[n], \quad \forall n \in \mathbb{Z}$$

where $N$ is called a *period*. The *smallest* such $N$ is called the *fundamental period* and is usually denoted as $N_0$. The *fundamental frequency* is defined as

$$\Omega_0 := \frac{2\pi}{N_0}$$

If $N_0$ is the fundamental period of $x[n]$, then any positive *integer* multiple $N = kN_0$ is a period of $x[n]$.

The real (complex) sinusoid $x[n] = \cos(\Omega n + \theta)$ $(x[n] = e^{j(\Omega n + \theta)})$ is periodic if and only if $|\Omega|$ is a *rational* multiple of $\pi$ of the form

$$|\Omega| = \frac{k}{N_0} 2\pi.$$

Furthermore, if $k$ and $N_0$ are *co-prime*, the fundamental period of the signal is given by $N_0$.

Examples:

- $x[n] = \cos(n)$.
- $x[n] = \cos\left(\frac{\pi}{2} n\right)$.
- $x[n] = \cos\left(\frac{3\pi}{2} n\right)$.
- $x[n] = \cos(2\pi n)$.

Combining periodic DT signals:

- Combining periodic DT signals always results in a periodic DT signal. A period of the combined signal is given by the lowest common multiple (LCM) of the periods of the individual signals.

- It is possible that the fundamental period of the overall signal is smaller than the LCM of the fundamental periods of the individual signals.

# 2 Discrete Time Fourier Series

Let $x[n]$ be a periodic DT signal with fundamental period $N_0$. Similar to CT signals, we want to write $x[n]$ as a linear combination of complex exponentials, e.g.

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 n},$$

where $\phi_k[n] = e^{jk\Omega_0 n}$ are periodic complex sinusoids for all integer $k$.

The key difference is that, the CT signals $\phi_k(t) = e^{jk\omega_0 t}$ are *distinct* for all values $k$. For DT signals, this is not the case. Since the time variable $n$ must be an integer, we have

$$\phi_{k+N_0}[n] = e^{j(k+N_0)\Omega_0 n} = e^{jk\Omega_0 n} \cdot e^{jN_0\Omega_0 n} = e^{jk\Omega_0 n} \cdot e^{j2\pi n} = e^{jk\Omega_0 n} = \phi_k[n]$$

Therefore, there are only $N_0$ distinct signals in the set $\{\phi_k[n] \mid k \in \mathbb{Z}\}$.

Implications: To uniquely determine the coefficients $a_k$, we need to specify $N_0$ different $\phi_k[n]$. In theory, we can pick *any* $N_0$ distinct $\phi_k[n]$. In practice, it is customary to choose $\phi_k[n]$ for $k = 0, 1, \ldots, N_0 - 1$. The sum in the Fourier series representation is thus written as

$$x[n] = \sum_{k=0}^{N_0-1} a_k e^{jk\Omega_0 n}.$$

Two questions:

1) What periodic DT signals will have such Fourier series representation?

2) How do we find $\{a_k\}$?

## 2.1 Answering Question 1

For any DT periodic signal with period $N$, there is a Fourier series representation. This can be seen by choosing $\Omega_0 = 2\pi/N$ writing the Fourier series representation for $n = 0, 1, \ldots, N-$

1 as

$$\begin{aligned}
x[0] &= \sum_{k=0}^{N-1} a_k \\
x[1] &= \sum_{k=0}^{N-1} a_k e^{jk\Omega_0} \\
x[2] &= \sum_{k=0}^{N-1} a_k e^{j2k\Omega_0} \\
&\vdots \\
x[N-1] &= \sum_{k=0}^{N-1} a_k e^{j(N-1)k\Omega_0}.
\end{aligned}$$

Note that this is a system of $N$ linear equations with $N$ unknowns $a_0, a_1, \ldots, a_{N-1}$. In the matrix-vector form, we have

$$\begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{j\Omega_0} & \cdots & e^{j(N-1)\Omega_0} \\ 1 & e^{j2\Omega_0} & \cdots & e^{j2(N-1)\Omega_0} \\ & & \vdots & \\ 1 & e^{j(N-1)\Omega_0} & \cdots & e^{j(N-1)(N-1)\Omega_0} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-1} \end{pmatrix}$$

The matrix is a scalar multiple of a unitary matrix (i.e., a complex rotation) and is known to be invertible for any positive integer $N$. Hence, given $x[0], x[1], \ldots, x[N-1]$, we can uniquely determine the Fourier series coefficients $a_0, a_1, \ldots, a_{N-1}$ by "inverting" the linear system equations.

Note that the answer for the DT case is very different from the CT case where only signals that satisfy the Dirichlet conditions will have Fourier series representations. The reason is that for the CT case, the summation in the synthesis equation is an *infinite* series, while for the DT case it is only over a *finite* number of terms.

## 2.2 Answering Question 2

Recall the finite geometric sum given by

$$\sum_{n=a}^{b} \alpha^n = \begin{cases} b-a+1, & \alpha = 1 \\ \frac{1}{1-\alpha}\left(\alpha^a - \alpha^{b+1}\right), & \alpha \neq 1. \end{cases}$$

Let $\alpha = e^{jk\Omega_0} = e^{jk2\pi/N}$. Then, $\alpha = 1$ when $k$ is an *integer* multiple of $N$ and $\alpha \neq 1$ otherwise. Thus, we have

$$\sum_{n=0}^{N-1} e^{jk\Omega_0 n} = \begin{cases} N, & \text{if } k \text{ is an integer multiple of } N \\ \frac{1-e^{jk\Omega_0 N}}{1-e^{jk\Omega_0}} = \frac{1-e^{jk2\pi}}{1-e^{jk\Omega_0}} = 0, & \text{otherwise.} \end{cases}$$

Now, we assume that

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n}.$$

Then, we multiply both sides by $e^{-jm\Omega_0 n}$, for some fixed $m \in \{0, \ldots, N-1\}$, and sum over $n = 0, \ldots, N-1$. This gives

$$\begin{aligned}\sum_{n=0}^{N-1} x[n] e^{-jm\Omega_0 n} &= \sum_{n=0}^{N-1} \left( \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n} \right) e^{-jm\Omega_0 n} \\ &= \sum_{k=0}^{N-1} a_k \left( \sum_{n=0}^{N-1} e^{j(k-m)\Omega_0 n} \right)\end{aligned}$$

Note that

$$\sum_{n=0}^{N-1} e^{j(k-m)\Omega_0 n} = \begin{cases} N, & \text{if } k = m \\ 0, & \text{if } k = 0, \ldots, m-1, m+1, \ldots, N-1 \end{cases}$$

We thus have

$$\sum_{n=0}^{N-1} x[n] e^{-jm\Omega_0 n} = N a_m$$

giving

$$\begin{aligned}a_m &= \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jm\Omega_0 n} \\ &= \frac{1}{N} \sum_{n \in \langle N \rangle} x[n] e^{-jm\Omega_0 n}\end{aligned}$$

for $m = 0, \ldots, N-1$, where $\langle N \rangle$ denotes any set of $N$ consecutive integers.

Periodic extension of $a_k$: As mentioned previously, for DT signals $\phi_k[n] = e^{jk\Omega_0 n}$, $\phi_{k+N}[n] = \phi_k[n]$. Therefore, analysis equation above implies that $a_{k+N} = a_k$ and one can think of $a_k$ as a periodic sequence with period $N$. In comparison, CTFS coefficients are generally aperiodic.

With this periodic extension of $a_k$, we have the following general DTFS pair (for any periodic $x[n]$ with period $N$):

$$x[n] = \sum_{k \in \langle N \rangle} a_k e^{jk\Omega_0 n} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{N} \sum_{n \in \langle N \rangle} x[n] e^{-jk\Omega_0 n} \qquad \text{(Analysis equation)}$$

# 3 Examples

Example 1: Assume that $N \geq 2N_1 + 1$ and let $x[n]$ be the periodic signal (with period $N$) given by

$$x[n] = \begin{cases} 1 & \text{if } 0 \leq n \leq N_1 \\ 1 & \text{if } N - N_1 \leq n < N \\ 0 & \text{otherwise.} \end{cases}$$

Find the FS coefficients of the periodic DT square wave $x[n]$.

Answer: By the analysis equation,

$$\begin{aligned} a_k &= \frac{1}{N} \sum_{n \in \langle N \rangle} x[n] e^{-jk\Omega_0 n} \\ &= \frac{1}{N} \sum_{n=-N_1}^{N_1} \left(e^{-jk\Omega_0}\right)^n. \end{aligned}$$

When $k$ is an integer multiple of $N$, we have $e^{-jk\Omega_0} = e^{-sk(2\pi/N)} = 1$ and hence

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} 1 = \frac{2N_1 + 1}{N} = d,$$

where $d$ is the duty cycle of the periodic DT square wave. When $k$ is not an integer multiple of $N$, then we have $e^{-jk\Omega_0} \neq 1$ and we can use the geometric sum formula to get

$$\begin{aligned} a_k &= \frac{1}{N} \sum_{n=-N_1}^{N_1} \left(e^{-jk\Omega_0}\right)^n \\ &= \frac{1}{N} \frac{e^{jk\Omega_0 N_1} - e^{-jk\Omega_0(N_1+1)}}{1 - e^{-jk\Omega_0}} \\ &= \frac{1}{N} \frac{e^{jk\Omega_0 N_1} - e^{-jk\Omega_0(N_1+1)}}{1 - e^{-jk\Omega_0}} \frac{e^{jk\Omega_0/2}\, e^{-jk\Omega_0/2}}{e^{jk\Omega_0/2}\, e^{-jk\Omega_0/2}} \\ &= \frac{1}{N} \frac{e^{jk\Omega_0(N_1-1/2)} - e^{-jk\Omega_0(N_1+1/2)}}{e^{jk\Omega_0/2} - e^{-jk\Omega_0/2}} \frac{e^{-jk\Omega_0/2}}{e^{-jk\Omega_0/2}} \\ &= \frac{\sin(\pi k d)}{N \sin(\pi k/N)}. \end{aligned}$$

The 3rd and 4th steps make use of a valuable trick related to the following general observation

$$\begin{aligned} e^a - e^{-b} &= \left(e^a - e^{-b}\right) e^{(b-a)/2} e^{-(b-a)/2} \\ &= \left(e^{(a+b)/2} - e^{-(a+b)/2}\right) e^{-(b-a)/2} \\ &= 2j \sin((a+b)/2)) e^{-(b-a)/2}. \end{aligned}$$

# 4 Properties of DTFS

The first set of properties can be proven in a manner nearly identical to the analagous property for CTFS.

**Property 1** (Linearity)**.** *Let $x[n]$ and $y[n]$ be two periodic DT signals with the same period $N$. Suppose that $x[n] \overset{DTFS}{\longleftrightarrow} a_k$ and $y[n] \overset{DTFS}{\longleftrightarrow} b_k$. Then*

$$\alpha x[n] + \beta y[n] \overset{DTFS}{\longleftrightarrow} \alpha a_k + \beta b_k$$

**Property 2** (Time Reversal)**.** *Suppose that $x[n] \overset{DTFS}{\longleftrightarrow} a_k$. Then*

$$x[-n] \overset{DTFS}{\longleftrightarrow} a_{-k}$$

**Property 3** (Even and Odd Symmetry)**.** *Suppose that $x[n] \overset{DTFS}{\longleftrightarrow} a_k$. If $x[n]$ is even, then $a_k$ is also even, i.e., $a_{-k} = a_k$. If $x[n]$ is odd, then $a_k$ is also odd, i.e., $a_{-k} = -a_k$.*

**Property 4** (Conjugation)**.** *Suppose that $x[n] \overset{DTFS}{\longleftrightarrow} a_k$. Then*

$$x^*[n] \overset{DTFS}{\longleftrightarrow} a^*_{-k}$$

**Property 5** (Conjugate Symmetry)**.** *Suppose that $x[n] \overset{DTFS}{\longleftrightarrow} a_k$ and $x[n]$ is real. Then*

$$a_{-k} = a_k^*$$

**Remark 1.** *The DTFS pair, $x[n] \overset{DTFS}{\longleftrightarrow} a_k$, satisfies the following symmetry conditions:*

1. *If $x[n]$ is real, then $Re\{a_k\}$ is even, $Im\{a_k\}$ is odd, $|a_k|$ is even, and $\angle a_k$ is odd.*
2. *If $x[n]$ is real and even, then $a_k$ is real and even*
3. *If $x[n]$ is real and odd, then $a_k$ is purely imaginary and odd.*

**Property 6** (Time Shift)**.** *Suppose that $x[n] \overset{DTFS}{\longleftrightarrow} a_k$. Then*

$$x[n - n_0] \overset{DTFS}{\longleftrightarrow} a_k e^{-jk\Omega_0 n_0}$$

The next set of properties are similar to properties of the CTFS but some have important differences.

**Property 7** (Multiplication)**.** *Let $x[n]$ and $y[n]$ be two periodic DT signals with the same period $N$. Suppose that $x[n] \overset{DTFS}{\longleftrightarrow} a_k$ and $y[n] \overset{DTFS}{\longleftrightarrow} b_k$. Then*

$$x[n]y[n] \overset{DTFS}{\longleftrightarrow} N(a_k \circledast b_k)$$

*where*

$$a_k \circledast b_k := \frac{1}{N} \sum_{l \in \langle N \rangle} a_l b_{k-l}.$$

**Property 8** (Parseval's Relation)**.** *Let $x[n]$ be a periodic DT signal with period $N$. Suppose that $x[n] \overset{DTFS}{\longleftrightarrow} a_k$. Then*

$$\frac{1}{N}\sum_{n\in\langle N\rangle}|x[n]|^2 = \sum_{k\in\langle N\rangle}|a_k|^2.$$

**Property 9** (Periodic Convolution)**.** *Let $x[n]$ and $y[n]$ be two periodic DT signals with the same period $N$, and let $z[n] = x[n] \circledast y[n]$. Suppose that $x[n] \overset{DTFS}{\longleftrightarrow} a_k$ and $y[n] \overset{DTFS}{\longleftrightarrow} b_k$. Then*

$$z[n] \overset{DTFS}{\longleftrightarrow} a_k b_k.$$

**Property 10** (Duality)**.** *A very important property of the DTFS is that*

$$x[n] \overset{DTFS}{\longleftrightarrow} a_k = y[k] \overset{DTFS}{\longleftrightarrow} b_m = x[-m].$$

*In words, this means that computing the DTFS of the DTFS of a signal returns the original signal with a time reversal.*

*Proof.* Suppose $x[n]$ is periodic with period $N$ and assume that $x[n] \overset{DTFS}{\longleftrightarrow} a_k$ and $y[k] = a_k$ for all integer $k$. Since $a_k$ is periodic with period $N$, we can also compute the DTFS pair $y[k] \overset{DTFS}{\longleftrightarrow} b_m$. In this case, we find that

$$\begin{aligned}
b_m &= \frac{1}{N}\sum_{k\in\langle N\rangle} y[k]e^{-jm\Omega_0 k} \\
&= \frac{1}{N}\sum_{k\in\langle N\rangle}\left(\frac{1}{N}\sum_{n\in\langle N\rangle} x[n]e^{-jk\Omega_0 n}\right)e^{-jm\Omega_0 k} \\
&= \frac{1}{N}\sum_{n\in\langle N\rangle} x[n]\frac{1}{N}\sum_{k\in\langle N\rangle} e^{-jk\Omega_0(m+n)} \\
&= \frac{1}{N}\sum_{n\in\langle N\rangle} x[n]\delta[m+n] \\
&= x[-m].
\end{aligned}$$

$\square$