ECEN 314: Signals and Systems

Lecture Notes 11: The Continuous-Time Fourier Transform

Reading:

- Current: SSOW 4.0-4.5
- Next: SSOW 5.0-5.7

1 The Frequency Domain

Consider the CT complex exponential input $x(t) = e^{st}$, where $s = \sigma + j\omega$ is an arbitrary complex number. The exponential growth rate of this signal is determined by σ and the oscillation period is given by $2\pi/\omega$. For a CT LTI system with impulse response $h(t)$, the associated output is given by the CT convolution integral

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau}_{H(s)},$$

where the function $H(s)$ is called the transfer function of the system. This implies that the output signal is equal to the input signal multiplied by the complex constant $H(s)$, which depends explicitly on s and implicitly on the system impulse response $h(t)$.

In these notes, we we will see that $H(j\omega)$ is known as the *Fourier transform* of $h(t)$ and characterizes the system in the *frequency domain*. The Fourier transform greatly simplifies the solution of many problems because it has many nice properties. Later, we will see that $H(s)$ is known as the Laplace transform of $h(t)$ and is also very useful. To see one example of how the Fourier transform simplifies things, consider the series connection of two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$, with transfer functions $H_1(s)$ and $H_2(s)$. Let the output be

$$y(t) = x(t) * \underbrace{h_1(t) * h_2(t)}_{h_{12}(t)} = x(t) * h_{12}(t).$$

For the input $x(t) = e^{st}$, our previous analysis shows that

$$y(t) = (x(t) * h_1(t)) * h_2(t) = H_1(s)x(t) * h_2(t) = H_1(s)H_2(s)x(t)$$

and $y(t) = x(t) * h_{12}(t) = H_{12}(s)x(t)$. This implies that the impulse response $h_{12}(t)$ for the series connection of the two systems has the transfer function $H_{12}(s) = H_1(s)H_2(s)$. More generally, it implies that convolution in the time domain corresponds to pointwise multiplication in the frequency domain.

2 The CT Fourier Transform (CTFT)

For a signal $x(t)$, let the *continuous-time Fourier transform* $X(j\omega)$ be defined by

$$X(j\omega) \triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt. \tag{1}$$

For the Fourier transform $X(j\omega)$, the inverse Fourier transform is defined by

$$x(t) \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega. \tag{2}$$

One fundamental question is, for what signals, does the Fourier transform $X(j\omega)$ exist (i.e., the integral well-defined) and satisfy 2. While this is an important question, we will save it for a little later.

First, we describe the CT Fourier transform as a natural limit of the CT Fourier series. Let $x(t)$ be a signal with finite energy that is supported on $[-A, A]$ (i.e., $x(t) = 0$ for $|t| > A$) and consider its periodic extension

$$x_T(t) \triangleq \sum_{k=-\infty}^{\infty} x(t - kT).$$

For all $T \geq 2A$, the function $x_T(t)$ consists of infinitely-many non-overlapping shifted copies of $x(t)$. Therefore, for $T \geq 2A$, the CT Fourier series of $x_T(t)$ is given by

$$\begin{aligned} a_k^{(T)} &= \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-jk(2\pi/T)t}dt \\ &= \frac{1}{T}\int_{-A}^{A} x(t)e^{-jk(2\pi/T)t}dt \\ &= \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk(2\pi/T)t}dt \\ &= \frac{1}{T}X(j\omega)\big|_{\omega=2\pi k/T}. \end{aligned}$$

This means that the Fourier series coefficients of $x_T(t)$ can be see seen as scaled samples of the Fourier transform, $X(j\omega)$, of $x(t)$. Since the Fourier series converges under these conditions, it follows that

$$x_T(t) = \sum_{k=-\infty}^{\infty} a_k^{(T)} e^{jk(2\pi/T)t}.$$

Using this, one can derive the inverse Fourier transform with

$$\begin{aligned}
x(t) &= \lim_{T\to\infty} x_T(t) \\
&= \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} a_k^{(T)} e^{jk(2\pi/T)t} \\
&= \lim_{T\to\infty} \frac{1}{T} \sum_{k=-\infty}^{\infty} T a_k^{(T)} e^{jk(2\pi/T)t} \\
&= \lim_{T\to\infty} \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j2\pi k/T) e^{j(2\pi k/T)t} \\
&= \int_{-\infty}^{\infty} X(j2\pi f) e^{j2\pi f t} dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega.
\end{aligned}$$

Now, we consider some examples.

Example 1. Consider the signal

$$x(t) = e^{-at}u(t), \quad a > 0.$$

For this signal, we can compute

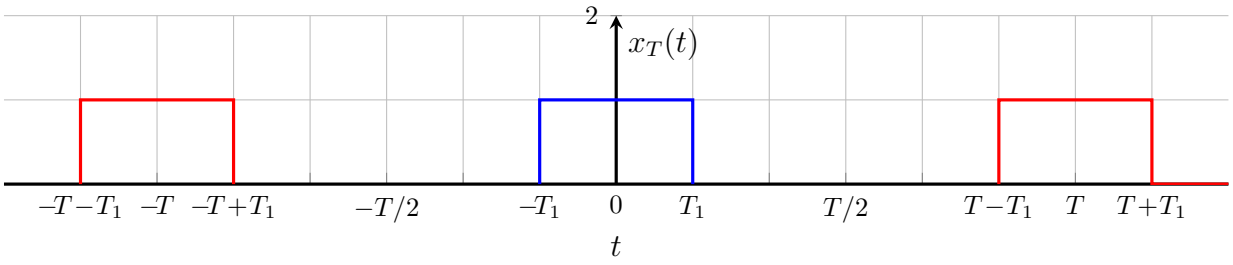
$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt \\
&= \int_{0}^{\infty} e^{-(a+j\omega)t} dt \\
&= \frac{1}{-a - j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty} \\
&= \frac{1}{a + j\omega}.
\end{aligned}$$

Since the Fourier transform is complex valued, it is usually best to find the magnitude and phase. For this signal, we get

$$|X(j\omega)| = \sqrt{X(j\omega)\left(X(j\omega)\right)^*} = \sqrt{\frac{1}{a+j\omega} \cdot \frac{1}{a-j\omega}} = \frac{1}{\sqrt{a^2+\omega^2}}$$

and

$$\angle X(j\omega) = \angle(1) - \angle(a + j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$



Example 2. Consider the rectangular pulse (shown in blue with red periodic extension)

$$x(t) = \begin{cases} 1 & \text{if } |t| < T_1 \\ 0, & \text{if } |t| > T_1. \end{cases}$$

Computing the Fourier transform gives

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)dt = \int_{-T_1}^{T_1} e^{-j\omega t}dt = -\frac{1}{j\omega}e^{-j\omega t}\Big|_{-T_1}^{T_1} = \frac{2\sin(\omega T_1)}{\omega}.$$

As we saw earlier, the CTFS are given by sampling this function at $\omega = 2\pi k/T$. Therefore, we have

$$X\left(j\frac{2\pi k}{T}\right) = \frac{2\sin(2\pi k T_1/T)}{2\pi k/T} = T\frac{\sin(\pi k d)}{\pi k} = Ta_k^{(T)},$$

where the CTFS coefficients of the periodic extension $x_T(t)$ are given by $\{a_k^{(T)}\}$.

Example 3. Consider the rectangular pulse

$$X(j\omega) = \begin{cases} 1 & \text{if } |\omega| < W \\ 0, & \text{if } |\omega| > W. \end{cases}$$

This gives

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}dt = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega = \frac{1}{2\pi jt}e^{j\omega t}\Big|_{-W}^{W} = \frac{\sin(Wt)}{\pi t}.$$

The similarity with the previous example is evidence of the *duality property* of Fourer transforms. Namely, that the Fourier transform of the Fourier transform results only in a scale factor and a time reversal.

Example 4. Now, consider the shifted Dirac delta $x(t) = \delta(t - t_0)$. This gives

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} \delta(t - t_0)e^{-j\omega t}dt \\ &= e^{-j\omega t_0}. \end{aligned}$$

Clearly, for $t_0 = 0$, this gives $X(j\omega) = 1$. By symmetry, this also implies that the inverse Fourier transform of $X(j\omega) = e^{-j\omega t_0}$ is $\delta(t - t_0)$. Due to mathematical subtleties, however, this result does not follow directly from the inverse Fourier transform integral.

Example 5. Likewise, we can consider the shifted Dirac delta $X(j\omega) = \delta(\omega - \omega_0)$ in the frequency domain. This gives

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)e^{j\omega t}dt \\ &= \frac{1}{2\pi}e^{j\omega_0 t}. \end{aligned}$$

Invoking symmetry again, this implies that the Fourier transform of $\frac{1}{2\pi}e^{j\omega_0 t}$ equals $\delta(\omega-\omega_0)$.

3 Convergence of the Fourier Transform

Since the Fourier transform is an improper integral, we must also consider convergence issues. In particular, we must require that

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \lim_{A,B\to\infty}\int_{-A}^{B} x(t)e^{-j\omega t}dt$$

is reasonably well-defined. Ideally, $x(t)$ is finite for all t and $X(j\omega)$ exists and is finite for all ω. But, this precludes the use of Dirac delta functions and other useful tools. Therefore, we are also interested in weaker conditions. A useful definition for engineering is that, in the limit, there is no *energy* in the difference signal

$$e_N(t) \triangleq x(t) - \frac{1}{2\pi}\int_{-N}^{N} X(j\omega)e^{j\omega t}dt.$$

In particular, this requires that

$$\lim_{N\to\infty}\int_{-\infty}^{\infty} |e_N(t)|^2\,dt = \lim_{N\to\infty}\int_{-\infty}^{\infty}\left|x(t) - \frac{1}{2\pi}\int_{-N}^{N} X(j\omega)e^{j\omega t}dt\right|^2 dt = 0.$$

It can be shown that this occurs whenever the signal $x(t)$ has finite energy

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt < \infty.$$

However, this *does not* mean that $\sup_{t\in\mathbb{R}} e_N(t)$ converges to 0 (e.g., Gibb's phenomenon implies this is bounded away from 0 if $x(t)$ is not continuous).

For real-world signals that satisfy the following Dirichlet conditions, we find that $\lim_{N\to\infty} e_N(t) = 0$ where $x(t)$ is continuous. At points of discontinuity, the value of the inverse Fourier transform converges to the midpoint

$$\frac{x(t^-) + x(t^+)}{2}.$$

The Dirichlet conditions are:

1. $x(t)$ is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

2. $x(t)$ only has a finite number of maxima and minima within any finite interval
3. $x(t)$ only has a finite number of finite jump discontinuities within any finite interval

4 Properties of the CT Fourier Transform

Linearity. Let $x(t)$ and $y(t)$ be two CT signals and suppose that $x(t) \overset{CTFT}{\longleftrightarrow} X(j\omega)$ and $y(t) \overset{CTFS}{\longleftrightarrow} Y(j\omega)$. Then

$$\alpha x(t) + \beta y(t) \overset{CTFT}{\longleftrightarrow} \alpha X(j\omega) + \beta Y(j\omega)$$

Time Shift. Suppose that $x(t) \overset{CTFT}{\longleftrightarrow} X(j\omega)$. Then,

$$x(t - t_0) \overset{CTFT}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega).$$

Proof. Let $y(t) = x(t - t_0)$ and $y(t) \overset{CTFT}{\longleftrightarrow} Y(j\omega)$. By the analysis equation,

$$\begin{aligned} Y(j\omega) &= \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} x(t) e^{-j\omega(t-t_0)} dt \\ &= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= e^{-j\omega t_0} X(j\omega). \end{aligned}$$

□

Time Reversal. Suppose that $x(t) \overset{CTFT}{\longleftrightarrow} X(j\omega)$. Then,

$$x(-t) \overset{CTFS}{\longleftrightarrow} X(j\omega).$$

Proof. Let$y(t) = x(-t)$ and $y(t) \overset{CTFT}{\longleftrightarrow} Y(j\omega)$. By the analysis equation,

$$\begin{aligned} Y(j\omega) &= \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt \\ &= X(-j\omega). \end{aligned}$$

□

Even/Odd Symmetry. Suppose that $x(t) \overset{CTFT}{\longleftrightarrow} X(j\omega)$. If $x(t)$ is even, then $X(j\omega)$ is also even, i.e., $X(-j\omega) = X(j\omega)$. If $x(t)$ is odd, then $X(j\omega)$ is also odd, i.e., $X(-j\omega) = -X(j\omega)$.

Proof. When $x(t)$ is even, we have $x(-t) = x(t)$. By the time-flip property,

$$X(-j\omega) = X(j\omega)$$

When $x(t)$ is odd, we have $x(-t) = -x(t)$. By the time-flip and linearity properties,

$$X(-j\omega) = -X(j\omega)$$

□

Conjugation. Suppose that $x(t) \overset{CTFT}{\longleftrightarrow} X(j\omega)$. Then

$$x^*(t) \overset{CTFT}{\longleftrightarrow} X^*(-j\omega)$$

Proof. Let $y(t) = x^*(t)$ and $y(t) \overset{CTFT}{\longleftrightarrow} Y(j\omega)$. By the analysis equation,

$$\begin{aligned} Y(j\omega) &= \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x^*(t)\left(e^{j\omega t}\right)^* dt \\ &= \int_{-\infty}^{\infty} \left(x(t)e^{j\omega t}\right)^* dt \\ &= \left(\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt\right)^* \\ &= X^*(-j\omega). \end{aligned}$$

□

Conjugate Symmetry. Recall that a CT signal $x(t)$ has *conjugate symmetry* if $x(-t) = x^*(t)$. Likewise, the Fourier transform, $X(j\omega)$, has *conjugate symmetry* if $X(-j\omega) = X^*(j\omega)$.

Suppose that $x(t) \overset{CTFT}{\longleftrightarrow} X(j\omega)$ and $x(t)$ is real. Then,

$$X(-j\omega) = X^*(j\omega).$$

Proof. If $x(t)$ is real, then $x^*(t) = x(t)$ and $x^*(t) \overset{CTFT}{\longleftrightarrow} X^*(-j\omega)$ implies

$$X(j\omega) = X^*(-j\omega).$$

□

$$\begin{aligned} a_k^* &= Re\{a_k\} - jIm\{a_k\} \\ &= |a_k|e^{-j\angle a_k} \end{aligned}$$

Remark 6. If $x(t)$ is real, then $Re\{X(j\omega)\}$ is even, $Im\{X(j\omega)\}$ is odd, $|X(j\omega)|$ is even, and $\angle X(j\omega)$ is odd. Combining Properties 2 and 4, we see that if $x(t)$ is real and even, then

$$X(-j\omega) = X^*(j\omega) = X(j\omega)$$

and $X(j\omega)$ is real and even. Likewise, if $x(t)$ is real and odd, then

$$X(-j\omega) = X^*(j\omega) = -X(j\omega)$$

and $X(j\omega)$ is pure imaginary and odd.

Frequency Shift. Suppose that $x(t) \overset{CTFT}{\longleftrightarrow} X(j\omega)$. Then, for any ω_0,

$$e^{j\omega_0 t}x(t) \overset{CTFT}{\longleftrightarrow} X(j(\omega - \omega_0)).$$

Proof. Let $e^{jM\omega_0 t}x(t) \overset{CTFS}{\longleftrightarrow} b_k$. By the analysis equation,

$$\begin{aligned} b_k &= \frac{1}{T}\int_0^T e^{jM\omega_0 t}x(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T}\int_0^T x(t)e^{-j(k-M)\omega_0 t}dt \\ &= a_{k-M}. \end{aligned}$$

□

Convolution. Let $x(t) \overset{CTFT}{\longleftrightarrow} X(j\omega)$ and $y(t) \overset{CTFT}{\longleftrightarrow} Y(j\omega)$ $x(t)$ be two Fourier transform pairs. Then,

$$x(t) * y(t) \overset{CTFT}{\longleftrightarrow} X(j\omega)Y(j\omega).$$

Proof. Applying the analysis equation to $z(t) = x(t) * y(t)$ gives

$$\begin{aligned}
Z(j\omega) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \right) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)y(t-\tau)e^{-j\omega(t-\tau)}e^{-j\omega\tau} d\tau dt \\
&= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau} \left(\int_{-\infty}^{\infty} y(t-\tau)e^{-j\omega(t-\tau)} dt \right) d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau} Y(j\omega) d\tau \\
&= X(j\omega)Y(j\omega).
\end{aligned}$$

□

Multiplication. Let $x(t) \overset{CTFT}{\longleftrightarrow} X(j\omega)$ and $y(t) \overset{CTFT}{\longleftrightarrow} Y(j\omega)$ $x(t)$ be two Fourier transform pairs. Then,

$$x(t)y(t) \overset{CTFT}{\longleftrightarrow} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta)Y(j(\omega-\theta))d\theta.$$

Proof. By the synthesis equation,

$$\begin{aligned}
x(t)y(t) &= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \right) \cdot \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\theta)e^{j\theta t} d\theta \right) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)Y(j\theta)e^{j(\omega+\theta)t} d\omega d\theta \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)Y(j(\phi-\omega)) d\omega \right) e^{j\phi t} d\phi.
\end{aligned}$$

Switching $\omega \mapsto \theta$ and $\phi \mapsto \omega$ completes the derivation because we see that $x(t)y(t)$ is the inverse Fourier transform of the convolution of $X(j\omega)$ and $Y(j\omega)$. □

Duality. For $x(t) \overset{CTFT}{\longleftrightarrow} X(j\omega)$, duality says that applying the Fourier transform twice returns $x(t)$ with a time reversal and scale factor. In particular, we have

$$\int_{-\infty}^{\infty} X(j\omega)e^{-j\tau\omega} d\omega = 2\pi x(-\tau).$$

Proof. Using an argument similar to that of the inverse Fourier transform, we can write

$$\begin{aligned}
2\pi x(-t) &= 2\pi \lim_{T\to\infty} x_T(-t) \\
&= 2\pi \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} a_k^{(T)} e^{-jk(2\pi/T)t} \\
&= 2\pi \lim_{T\to\infty} \frac{1}{T} \sum_{k=-\infty}^{\infty} T a_k^{(T)} e^{-jk(2\pi/T)t} \\
&= 2\pi \lim_{T\to\infty} \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j2\pi k/T) e^{-j(2\pi k/T)t} \\
&= 2\pi \int_{-\infty}^{\infty} X(j2\pi f) e^{-j2\pi ft} dt \\
&= \int_{-\infty}^{\infty} X(j\omega) e^{-j\omega t} d\omega.
\end{aligned}$$

□

Parseval's Relation. For $x(t) \overset{CTFT}{\longleftrightarrow} X(j\omega)$, we have

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega.$$

Remark 7. Note that $\int_{-\infty}^{\infty} |x(t)|^2 dt$ represents the total energy in $x(t)$ (measured in the time domain), and $\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$ represents the total energy in $x(t)$ (measured in the frequency domain). So Parseval's relation basically states the energy in $x(t)$ is the same whether it is measured in the time or frequency domain.

Proof. Let $x(t) \overset{CTFT}{\longleftrightarrow} X(j\omega)$, $y(t) = |x(t)|^2 = x(t)x^*(t)$, and $y(t) \overset{CTFT}{\longleftrightarrow} Y(j\omega)$. First, we calculate directly

$$Y(0) = Y(j\omega)\big|_{\omega=0} = \int_{-\infty}^{\infty} y(t) dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

Next, we use the conjugaton property and the mutiplication property to write

$$Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(-j\theta) X(j(\omega - \theta)) d\theta.$$

For $\omega = 0$, this gives

$$\begin{aligned}
Y(0) &= Y(j\omega)\big|_{\omega=0} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(-j\theta) X(-j\theta) d\theta \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\theta)|^2 \, d\theta.
\end{aligned}$$

Combining these two results completes the proof. □

Differentiation. Suppose that $x(t) \overset{CTFT}{\longleftrightarrow} X(j\omega)$. Then,

$$\frac{d}{dt}x(t) \overset{CTFT}{\longleftrightarrow} j\omega X(j\omega)$$

In particular, the DC component of $dx(t)/dt$ is always equal to zero.

Proof. By the synthesis equation,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

Taking derivative on both sides gives

$$\begin{aligned}\frac{d}{dt}x(t) &= \frac{d}{dt}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\ &= j\omega\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\ &= j\omega X(j\omega).\end{aligned}$$

□

Integration. Suppose that $x(t) \overset{CTFT}{\longleftrightarrow} X(j\omega)$. Then,

$$\int_{-\infty}^{t} x(t)dt \overset{CTFT}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega).$$

In this case, a DC component in $x(t)$ results in a Dirac delta function at $\omega = 0$ in $X(j\omega)$.