ECEN 314: Signals and Systems

Lecture Notes 3: Basic System Properties

Reading:

- Current: SSOW 1.5-1.6
- Next: SSOW 2.1

1 Causality

A system is *causal* if the output does not anticipate future values of the input, i.e., if the output at any time depends only on values of the input up to that time.

- All real-time physical systems are causal, because time only moves forward. Effect occurs after cause. (Imagine if you own a non-causal system whose output depends on tomorrows stock price.)
- Causality does not apply to spatially varying signals. (We can move both left and right, up and down.)
- Causality does not apply to systems processing recorded data.

Example (Causal or non-causal):

- $y(t) = x^2(t-1)$
- $y(t) = x(t+1)$
- $y[n] = x[-n]$
- $y[n] = 2^{n+1}x^3[n-1]$

Mathematically (in CT), consider a system $x(t) \to y(t)$. Let $x_1(t)$ and $x_2(t)$ be two input signals with corresponding output signals $y_1(t)$ and $y_2(t)$, respectively. Then, the system is causal if and only if

$$x_1(t) = x_2(t), \quad \forall t < t_0 \quad \Longrightarrow \quad y_1(t) = y_2(t), \quad \forall t < t_0$$

2 Time-invariance

Informally, a system is *time-invariant (TI)* if its behavior does not depend on what time it is. Mathematically (in CT), a system $x(t) \to y(t)$ is TI if

$$x(t) \to y(t) \quad \Longrightarrow \quad x(t-t_0) \to y(t-t_0), \quad \forall t_0 \in \mathbb{R}$$

Example (TI or time-varying (TV)):

- $y(t) = x^2(t-1)$. To check whether this system is TI or TV, we need to determine the output corresponding to the input $x(t-t_0)$ and then compare it with $y(t-t_0)$. Let $x'(t) = x(t-t_0)$. Then $y'(t) = [x'(t-1)]^2 = [x(t-1-t_0)]^2$. On the other hand, $y(t-t_0) = x^2(t-t_0-1)$. We conclude that $y'(t) = y(t-t_0)$ so the system is TI.

- $y[n] = 2^{n+1}x^3[n-1]$. Let $x'[n] = x[n-n_0]$. Then, $y'[n] = 2^{n+1}(x'[n-1])^3 = 2^{n+1}(x[n-n_0-1])^3$. On the other hand, $y[n-n_0] = 2^{n-n_0+1}x^3[n-n_0-1]$. In general, $y'[n] \neq y[n-n_0]$, so the system is TV.

- $y[n] = x[-n]$. Let $x'[n] = x[n-n_0]$. Then, $y'[n] = x'[-n] = x[-n-n_0]$. On the other hand, $y[n-n_0] = x[-(n-n_0)] = x[-n+n_0]$. In general, $y'[n] \neq y[n-n_0]$, so the system is TV.

Fact: If the input to a TI system is periodic with a period T, then the output is also periodic with a period T.

Proof. Suppose $x(t+T) = x(t)$ for all $t \in \mathbb{R}$ and $x(t) \to y(t)$. Then, by TI, $x(t+T) \to y(t+T)$. The output of a system is fully determined by the input, so $y(t) = y(t+T)$, i.e., the output is also periodic with the same period T. $\square$

3 Linearity

A (CT) system is *linear* if it satisfies the following property:

$$x_1(t) \to y_1(t) \quad \text{and} \quad x_2(t) \to y_2(t) \quad \Longrightarrow \quad ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t), \quad \forall a, b \in \mathbb{C}$$

Properties: Let $x(t) \to y(t)$ be a linear CT system. Then,

- (Additivity)

$$x_1(t) \to y_1(t) \quad \text{and} \quad x_2(t) \to y_2(t) \quad \Longrightarrow \quad x_1(t) + x_2(t) \to y_1(t) + y_2(t)$$

- (Homogeneity)

$$x_1(t) \to y_1(t) \quad \Longrightarrow \quad ax_1(t) \to ay_1(t), \quad \forall a \in \mathbb{C}$$

- (Superposition)

$$x_k(t) \to y_k(t) \quad \Longrightarrow \quad \sum_k a_k x_k(t) \to \sum_k a_k y_k(t), \quad \forall a_k \in \mathbb{C}$$

Example (Linear or nonlinear):

- $y(t) = x(2t)$. Let $y_1(t) = x_1(2t)$, $y_2(t) = x_2(2t)$, and $x'(t) = ax_1(t) + bx_2(t)$. Then $y'(t) = x'(2t) = ax_1(2t) + bx_2(2t) = ay_1(t) + by_2(t)$. Conclusion: Linear.
- $y[n] = x^2[n]$. Let $y[n] = x^2[n]$ and $x'[n] = 2x[n]$. Then $y'[n] = (x'[n])^2 = (2x[n])^2 = 4x^2[n] \neq 2y[n]$. Conclusion: The system is not additive and hence is nonlinear.

Facts:

a) A system is linear if and only if it is both additive and homogeneous.

b) For linear systems, zero input $\longrightarrow$ zero output.

c) A linear system is causal if and only if it satisfies the condition of initial rest:

$$x(t) = 0, \quad \forall t < t_0 \quad \Longrightarrow \quad y(t) = 0, \quad \forall t < t_0$$

Proof. Part a): We only need to prove the "if" part. Suppose that the system $x(t) \to y(t)$ is both additive and homogeneous and that $x_1(t) \to y_1(t)$, $x_2(t) \to y_2(t)$. By homogeneity, $ax_1(t) \to ay_1(t)$ and $bx_2(t) \to by_2(t)$. Furthermore, by additivity, $ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$. Conclusion: The system is linear.

Part b): Let $x(t) \to y(t)$ be a linear system. By homogeneity, $0 \cdot x(t) \to 0 \cdot y(t)$. Conclusion: Zero input $\longrightarrow$ zero output.

Part c): Let $x(t) \to y(t)$ be a linear system. Let $x(t) = 0$ for all $t < t_0$ and let $x'(t) = 0$ be a zero input. From part b), $y'(t) = 0$ is a zero output. Note that $x(t) = x'(t)$ for all $t < t_0$. If the system is also causal, then $y(t) = y'(t) = 0$ for all $t < t_0$. That is, the initial rest condition must be satisfied. Conversely, suppose that the initial rest condition is satisfied. Let $x(t) = x'(t)$ for all $t < t_0$. Then, $x(t) - x'(t) = 0$ for all $t < t_0$. By linearity, $x(t) - x'(t) \to y(t) - y'(t)$. Furthermore, by the initial rest condition, the output $y(t) - y'(t) = 0$ for all $t < t_0$. Thus, we have $y(t) = y'(t)$ for all $t < t_0$, and we conclude that the system must be causal.

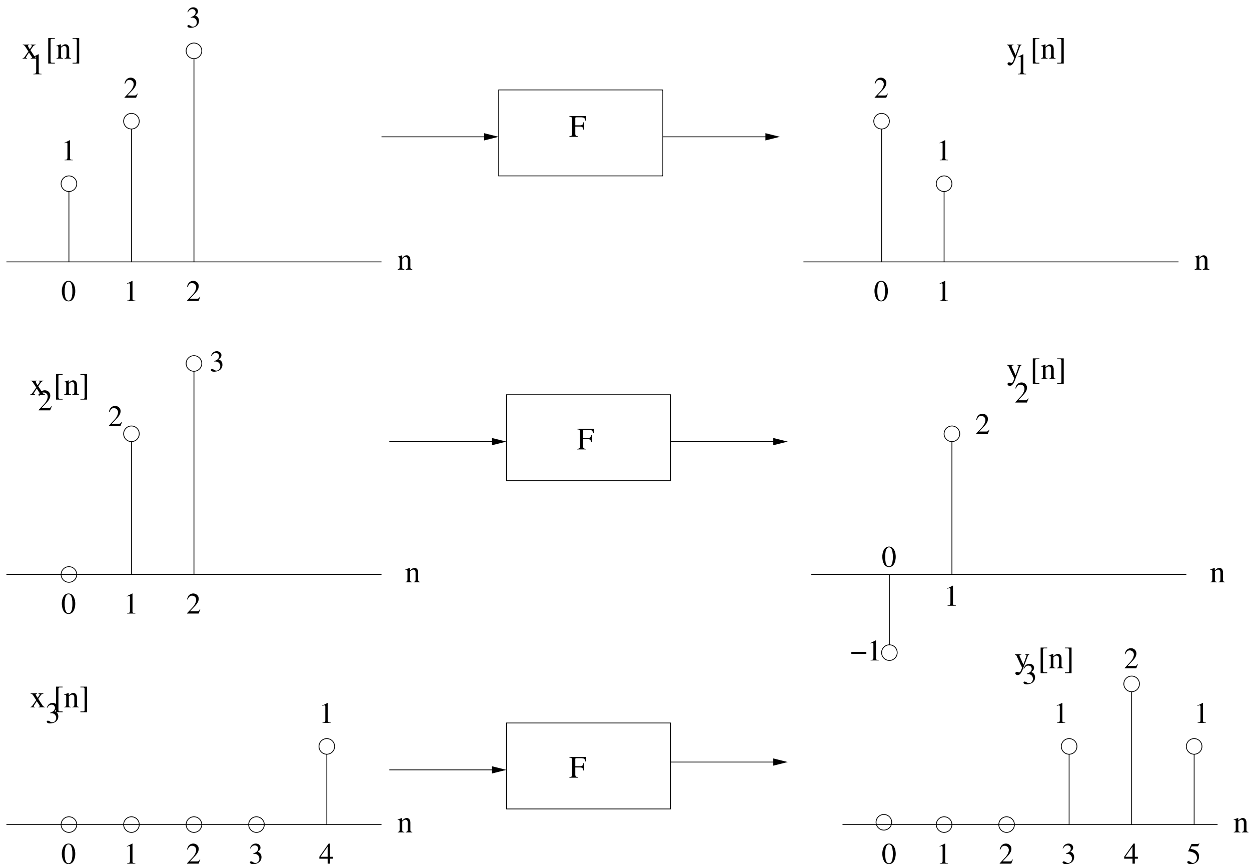
□

Exercise: A DT system F is known to be TI. Three test input-output signal pairs are depicted below (all signals are zero outside the region shown).

(a) Could the system be causal?

(b) Determine the response of the system to the unit impulse input.

(c) Could the system be linear?



$x_1[n]$
F
$y_1[n]$
$x_2[n]$
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$y_2[n]$
$x_3[n]$
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