ECEN 314: Signals and Systems

Lecture Notes 4: Discrete-Time Convolution

Reading:

- Current: SSOW 2.1
- Next: SSOW 2.2

1 LTI systems

Recall that a DT system is time invariant (TI) if

$$x[n] \to y[n] \quad \Longrightarrow \quad x[n-n_0] \to y[n-n_0], \qquad \text{for all integer } n_0$$

and that it is linear if

$$x_1[n] \to y_1[n] \quad \text{and} \quad x_2[n] \to y_2[n] \quad \Longrightarrow \quad ax_1[n]+bx_2[n] \to ay_1[n]+by_2[n], \qquad \text{for all complex } a, b$$

For this class, we focus on systems that are both linear and time invariant (LTI) due to:

- practical importance; and
- the powerful analysis tools associated with LTI systems.

A basic fact: If we know the response of an LTI system to some inputs, then we actually know the response to many inputs.

Question: What is the *smallest* set of inputs for which, if we know their outputs, we can determine the output of *any* input signal?

2 DT Convolution

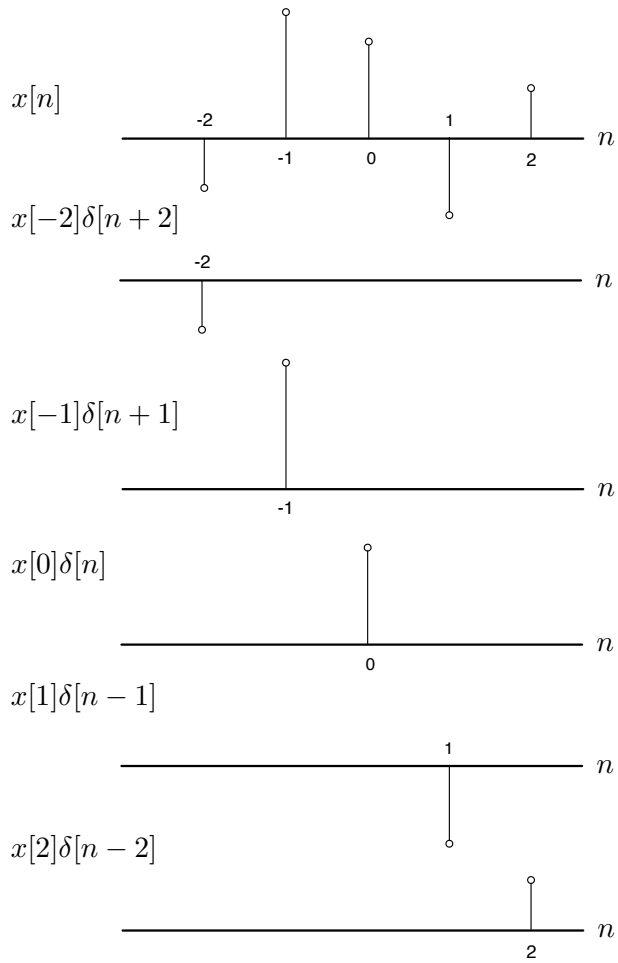
For DT systems, the answer is surprisingly simple: All we need to know is the *impulse response* (denoted by $h[n]$) which is the response to a unit impulse input

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

The reason for this is that we can write any signal $x[n]$ as a linear combination of the unit impulse function and its time-shifts:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

where $x[k]$ are coefficients and $\delta[n-k]$ is a time shift of $\delta[n]$.



Let $y[n] = h_k[n]$ be the system response to the input $x[n] = \delta[n-k]$. By linearity,

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

Furthermore, by TI,

$$\delta[n] \to h[n] \quad \Longrightarrow \quad \delta[n-k] \to h_k[n] = h[n-k]$$

The surprising conclusion is that the output of an LTI system is given by the "convolution" sum

$$y[n] = x[n] * h[n] \triangleq \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Observation: If we know the unit impulse response $h[n]$ of a LTI system, we can compute the output $y[n]$ of an arbitrary input $x[n]$ as $y[n] = x[n] * h[n]$. In this sense, a LTI system is fully determined by its unit impulse response.

Visualizing the calculation of convolution sum:

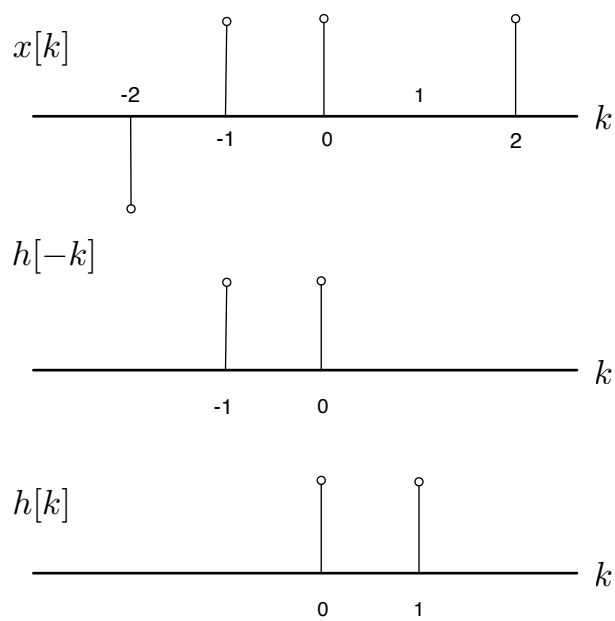
Step 1: Choose a value of n and consider it fixed.

Step 2: Plot $x[k]$ as a function of k.

Step 3: Plot the function $h[n-k]$ (as a function of k) by first flipping $h[k]$ and then shift to the right by n (if n is negative, this means a shift to the left by $|n|$.).

Step 4: Compute the intermediate signal $w_n[k] \triangleq x[k]h[n-k]$ via pointwise multiplication and then sum this signal to obtain the result $y[n]$.

To compute $y[n+1]$, one can compute $h[n+1-k]$ simply by shifting $h[n-k]$ to the right by sample. Then, answer is computed by repeating Step 4.



Example: Consider the DT system described by the linear constant-coefficient difference equation (LCCDE)

$$y[n] - \frac{1}{2}y[n-1] = x[n].$$

We assume the system is initially at rest, which is defined mathematically as

$$x[k] = 0 \text{ for all integer } k \leq k_0 \quad \Longrightarrow \quad y[k] = 0 \text{ for all integer } k \leq k_0.$$

It turns out that a system is LTI if it is described by a LCCDE and it is initially at rest. In this case, we can first figure out the unit impulse response of the system $h[n]$ and then use the convolution sum to calculate the response to $u[n]$. It is not too hard to verify that $y[n] = \left(\frac{1}{2}\right)^n u[n]$ satisfies the above LCCDE for input $x[n] = \delta[n]$. Therefore, we say that its impulse response is

$$h[n] = \left(\frac{1}{2}\right)^n u[n].$$

Now, we can calculate the output associated with the input $x[n] = u[n]$, which is known as the *step response* of the system. By the convolution sum, the output $y[n]$ corresponding to the input $x[n] = u[n]$ can be calculated as

$$\begin{aligned}
y[0] &= 1 \\
y[1] &= \frac{1}{2} + 1 = \frac{3}{2} \\
y[2] &= \left(\frac{1}{2}\right)^2 + \frac{1}{2} + 1 = \frac{7}{4} \\
&\vdots \\
y[n] &= \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} + \ldots + \frac{1}{2} + 1 = \frac{1 - \frac{1}{2}\left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 2 - \left(\frac{1}{2}\right)^n
\end{aligned}$$

for $n \geq 0$ and $h[n] = 0$ for $n \leq -1$.