ECEN 314: Signals and Systems

Lecture Notes 5: Properties of DT Convolution

Reading:

- Current: SSOW 2.3
- Next: SSOW 2.4

1 Properties of Convolution

Property 1 (The sifting property)**.** *Convolution with a time-shifted impulse results in a time-shifted version of the output:*

$$x[n] * \delta[n-n_0] = x[n-n_0].$$

Therefore, $x[n] * \delta[n] = x[n]$ *and* $\delta[n]$ *is the identity element under convolution.*

Proof. By definition,

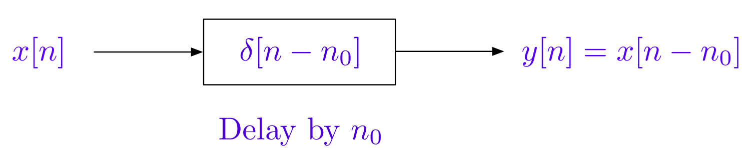
$$x[n] * \delta[n-n_0] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k-n_0]$$

and $x[k]\delta[n-k-n_0] = x[n-n_0]\delta[n-k-n_0]$ because

$$\delta[n-k-n_0] = \begin{cases} 1, & k = n-n_0 \\ 0, & k \neq n-n_0 \end{cases}$$

Computing the sum shows that $y[n] = x[n-n_0]$ and therefore $x[n]*\delta[n-n_0] = x[n-n_0]$. □

We can interpret this property using the following block diagram:



Property 2 (The commutative property)**.** *The convolution operator is commutative:*

$$y[n] = x[n] * h[n] = h[n] * x[n].$$

Proof. By definition,

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

and

$$h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

For the first, sum we can substitute $k = n - k'$ to get

$$x[n] * h[n] = \sum_{k'=-\infty}^{\infty} x[n-k']h[k'],$$

where the limits of summation are unchanged because they are infinite. We conclude that $x[n] * h[n] = h[n] * x[n]$. □

Property 3 (The distributive property)**.** *Convolution distributes over addition:*

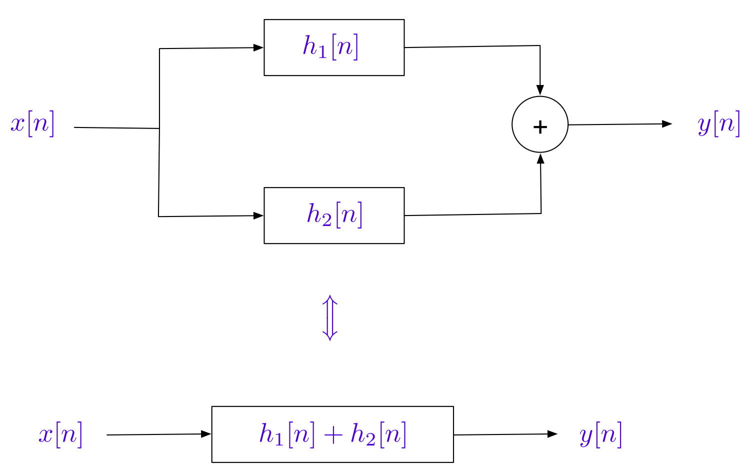
$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

Proof. By definition,

$$\begin{aligned} x[n] * (h_1[n] + h_2[n]) &= \sum_{k=-\infty}^{\infty} x[k](h_1[n-k] + h_2[n-k]) \\ &= \sum_{k=-\infty}^{\infty} x[k]h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k]h_2[n-k] \\ &= x[n] * h_1[n] + x[n] * h_2[n] \end{aligned}$$

□

Interpretation:



Property 4 (The associative property)**.** $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$

Proof. Let

$$a[n] = h_1[n] * h_2[n] = \sum_{k=-\infty}^{\infty} h_1[k]h_2[n-k]$$

and

$$b[n] = x[n] * h_1[n] = \sum_{l=-\infty}^{\infty} x[l]h_1[n-l]$$

Then

$$\begin{aligned} x[n] * a[n] &= \sum_{l=-\infty}^{\infty} x[l]a[n-l] \\ &= \sum_{l=-\infty}^{\infty} x[l] \left(\sum_{k=-\infty}^{\infty} h_1[k]h_2[n-l-k] \right) \end{aligned}$$

and

$$\begin{aligned} b[n] * h_2[n] &= \sum_{k=-\infty}^{\infty} b[k]h_2[n-k] \\ &= \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{\infty} x[l]h_1[k-l] \right) h_2[n-k] \\ &= \sum_{l=-\infty}^{\infty} x[l] \left(\sum_{k=-\infty}^{\infty} h_1[k-l]h_2[n-k] \right) \end{aligned}$$

Let $k' = k - l$. We have $k = l + k'$ and hence

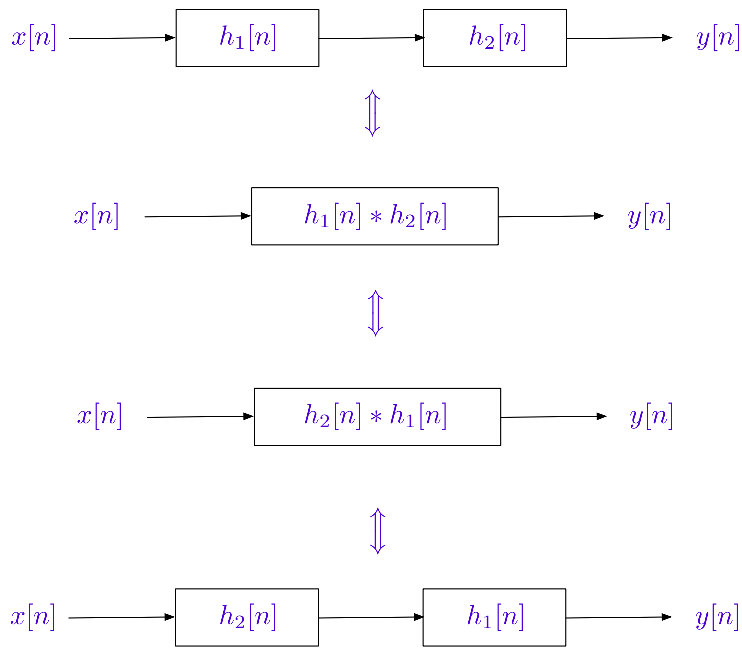
$$b[n] * h_2[n] = \sum_{l=-\infty}^{\infty} x[l] \left(\sum_{k'=-\infty}^{\infty} h_1[k']h_2[n-l-k'] \right)$$

Note that k and k' are indices of summation and, therefore, we conclude that $x[n] * a[n] = b[n] * h_2[n]$. □

Combining properties 2 and 4, we have

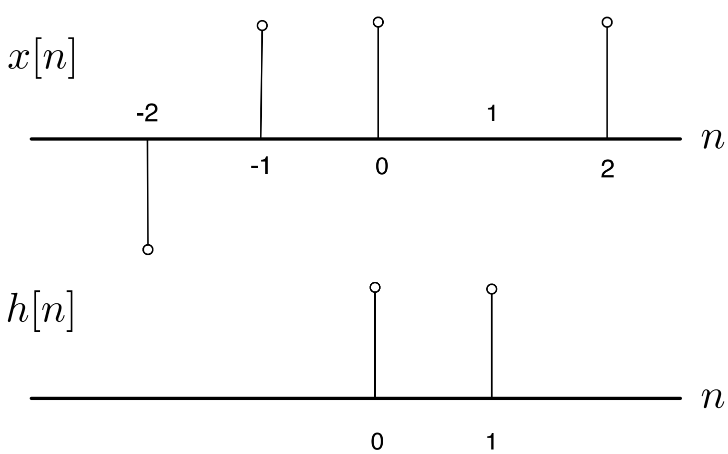
$$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n]) = x[n] * (h_2[n] * h_1[n]) = (x[n] * h_2[n]) * h_1[n]$$

Interpretation:



Combining properties 1 and 3, one can compute the convolution of two finite signals algebraically.

Example 1. *Consider the signals*



and observe that

$$x[n] = -\delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-2]$$

and

$$h[n] = \delta[n] + \delta[n-1].$$

Thus,

$$\begin{aligned}
x[n]*h[n] &= (-\delta[n+2]+\delta[n+1]+\delta[n]+\delta[n-2])*(\delta[n]+\delta[n-1]) \\
&= -\delta[n+2]+\delta[n+1]+\delta[n]+\delta[n-2]+ \\
&\qquad\qquad (-\delta[n+1]+\delta[n]+\delta[n-1]+\delta[n-3]) \\
&= -\delta[n+2]+2\delta[n]+\delta[n-1]+\delta[n-2]+\delta[n-3]
\end{aligned}$$

Property 5 (Causality)**.** *A DT LTI system is causal if and only if its unit impulse response* $h[n]=0$ *for all* $n<0$*.*

Proof. If a DT LTI system is causal, then the unit impulse response $h[n]=0$ for all $n<0$, as its input $\delta[n]=0$ for all $n<0$. Conversely, if the unit impulse response $h[n]=0$ for all $n<0$, by the convolution sum

$$y[n]=x[n]*h[n]=h[n]*x[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]=\sum_{k\geq 0}h[k]x[n-k]$$

i.e., $y[n]$ is fully determined by the current and previous inputs $x[n-k]$ for $k\geq 0$. We conclude that the system must be causal. □

Property 6 (Stability)**.** *A DT LTI system is stable if and only if* $\sum_{k=-\infty}^{\infty}|h[n]|<\infty$*.*

Proof. If $A=\sum_{k=-\infty}^{\infty}|h[n]|<\infty$ and the input $x[n]$ is bounded (i.e., $|x[n]|\leq M_x<\infty$ for all n), then

$$|y[n]|=\left|\sum_{k=-\infty}^{\infty}x[k]h[n-k]\right|\leq\sum_{k=-\infty}^{\infty}|x[k]||h[n-k]|\leq M_x\sum_{k=-\infty}^{\infty}|h[n-k]|=M_xA<\infty.$$

The proof of the "only if" part is outlined in SSOW problem 2.49. □

A final word: We said all along that a DT LTI system is fully characterized by its unit impulse response $h[n]$. However, if a system is not LTI to begin with, knowing the unit impulse response $h[n]$ may shed very little insight about what the system is. In fact, many systems may share the same unit impulse response $h[n]$. Consider, for example, $h[n]=\delta[n]$ is the unit impulse response to any system $y[n]=(x[n])^k$ with an integer k. But among them, there is only one system that is LTI. That is, $y[n]=x[n]$.