ECEN 314: Signals and Systems

Lecture Notes 6: Continuous-Time Convolution

Reading:

- Current: SSOW 2.2
- Next: SSOW 2.5

1 Continuous-time convolution

For DT LTI systems, we have

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \quad \Longrightarrow \quad y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

To achieve a similar result for CT LTI systems, we need to express $x(t)$ as a linear combination of a "basic" signal and its time shifts. The problem is that we need an uncountably infinite number of time shifts to cover the real line. Therefore, the decomposition must be in terms of an integral. Can we use

$$z(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0. \end{cases}$$

The problem with this is that $\int z(t)x(t)dt = 0$.

Instead, CT convolution can be derived by combining the sifting property of $\delta(t)$ with the linearity and time-invariance of the system. First, we observe that the sifting property allows us to write

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

This shows that any signal $x(t)$ can be be written as a weighted linear combination (via integration) of delta functions.

Next, we consider a linear system $x(t) \stackrel{S}{\to} y(t)$ and define $h(t;\tau)$ to be the output signal (i.e., function of t) from S associated with the input $\delta(t-\tau)$ (i.e., $\delta(t-\tau) \stackrel{S}{\to} h(t;\tau)$). We note that linearity of a system extends to integrals because integrals are defined to be limits of sums. Therefore, $x(t;\tau) \stackrel{S}{\to} y(t;\tau)$ for all $a \leq \tau \leq b$ implies

$$\int_a^b x(t;\tau)d\tau \stackrel{S}{\to} \int_a^b y(t;\tau)d\tau.$$

Combining this with the previous observation shows that

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \xrightarrow{S} y(t) = \int_{-\infty}^{\infty} x(\tau)h(t;\tau)d\tau.$$

Finally, the time-invariance of the system implies that

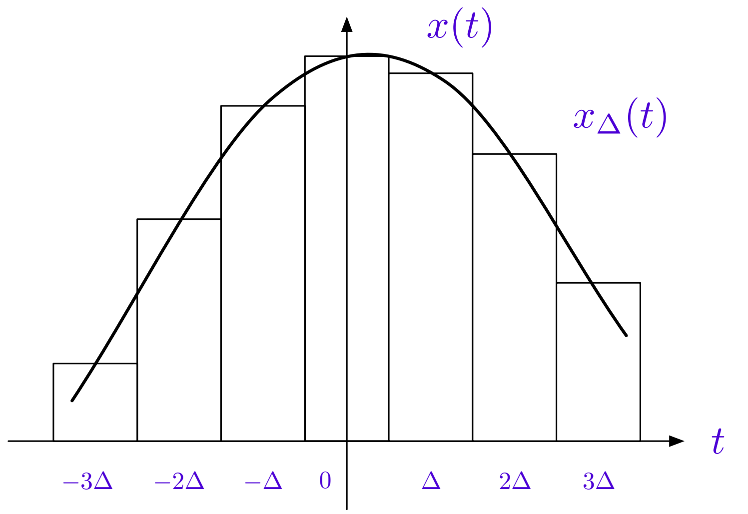
$$\delta(t-\tau) \xrightarrow{S} h(t-\tau;0)$$

and $h(t-\tau;0)$ is exactly the impulse response $h(t)$. Therefore, we find that

$$y(t) = x(t) * h(t) \triangleq \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

2 CT convolution via approximation

The linearity argument used above can be treated in more detail by considering the piecewise constant approximation of $x(t)$. For $x(t)$ and $\Delta > 0$, the piecewise constant approximation is $x_\Delta(t) = x(k\Delta)$ when $t \in ((k-1/2)\Delta, (k+1/2)\Delta]$. When $x(t)$ is continuous and $\Delta \to 0$, we have $x_\Delta(t) \to x(t)$ for all $t \in \mathbb{R}$.

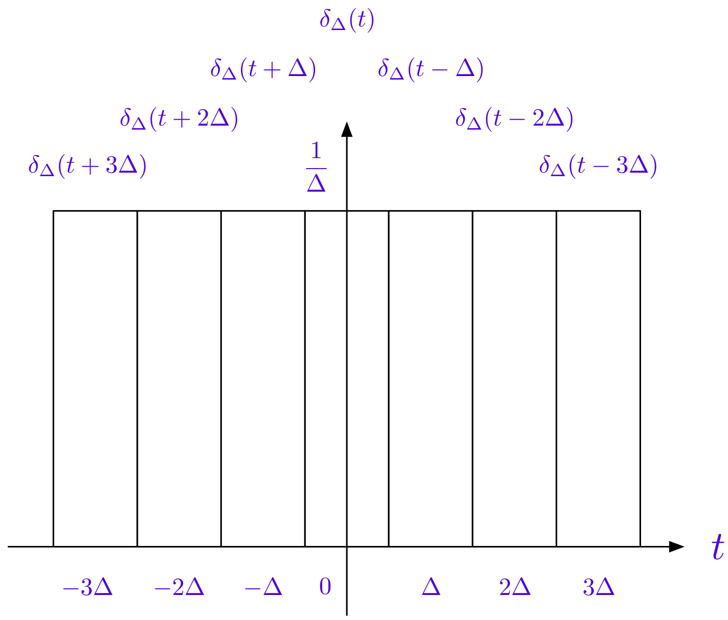


Can we express $x_\Delta(t)$ as a linear combination of a "basic" signal and its time shifts? Yes, consider

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & t \in \left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right] \\ 0, & \text{otherwise} \end{cases}$$

Note that $\delta_\Delta(t - k\Delta)$ are non-overlapping for different k and together they "fill up" the entire time axis. We thus have

$$x_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$



where Δ here is used to compensate $1/\Delta$ in the definition of $\delta_\Delta(t)$.

Let $h_\Delta(t)$ be the response to $\delta_\Delta(t)$. By the LTI properties, the output signal

$$y_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t-k\Delta)\Delta$$

When $\Delta \to 0$, let $\delta(t)$ and $h(t)$ be the limits of $\delta_\Delta(t)$ and $h_\Delta(t)$, respectively. Then,

$$x(t) \longleftarrow x_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta \longrightarrow \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

and

$$y(t) \longleftarrow y_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t-k\Delta)\Delta \longrightarrow \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

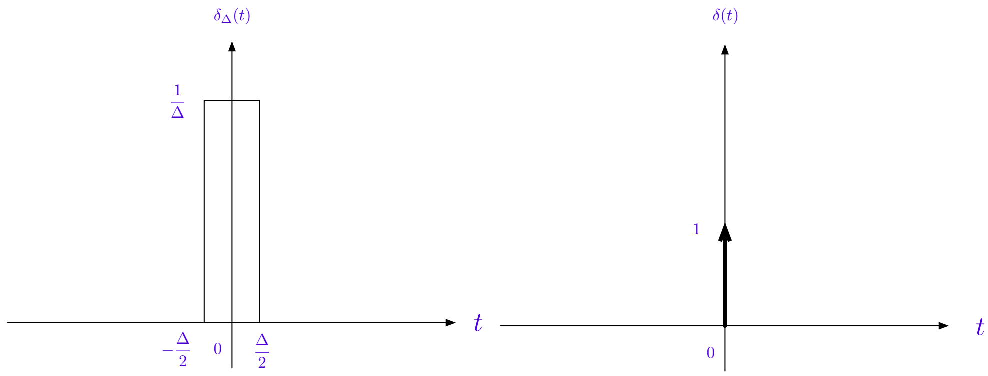
Conclusion, the convolution integral is given by

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t.)$$

But what is $h(t)$? Note that $\delta_\Delta(t) \to h_\Delta(t)$ so $\delta(t) \to h(t)$, i.e., $h(t)$ is the response to $\delta(t)$. But what is $\delta(t)$ anyway? Answer: It is a singularity function and it is operationally defined via the following properties:

1) $\delta(t) = 0$ for all $t \neq 0$; and

2) $\int_{0^-}^{0^+} \delta(t)dt = 1$.



Observation: If we know the unit impulse response $h(t)$ of a LTI system, we can compute the output $y(t)$ of an arbitrary input $x(t)$ as $y(t) = x(t) * h(t)$. In this sense, a LTI system is fully determined by its unit impulse response.

3 Properties of CT convolution

Property 1 (The sifting property)**.** $x(t)*\delta(t-t_0) = x(t-t_0)$. *In particular,* $x(t)*\delta(t) = x(t)$.

Proof. By definition, we have

$$\begin{aligned} x(t) * \delta(t - t_0) &= \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau - t_0)d\tau \\ &= x(t - t_0), \end{aligned}$$

where the integral is computed using the sifting proprty of $\delta(t)$. □

Property 2 (The commutative property)**.** $x(t) * h(t) = h(t) * x(t)$

Property 3 (The distributive property)**.** $x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$

Property 4 (The associative property)**.** $x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t)$

Property 5 (Causality)**.** *A CT LTI system is causal if and only if its unit impulse response* $h(t) = 0$ *for all* $t < 0$*.*

Property 6 (Stability)**.** *A CT LTI system is BIBO stable if and only if its unit impulse response satisfies* $\int_{-\infty}^{\infty} |h(t)|dt < \infty$*.*

Proofs of Properties 2–6 are completely analogous to those for DT systems and hence are omitted from this note.

4 Calculation of convolution integrals

Convolution integrals can be calculated using a similar graphical procedure as that for calculating convolution sums:

Step 1: Choose a value of t and consider it fixed.

Step 2: Plot $x(\tau)$ as a function of τ.

Step 3: Plot the function $h(t-\tau)$ (as a function of τ) by first flipping $h(\tau)$ and then shifting to the right by t (if t is negative, this means a shift to the left by $|t|$).

Step 4: Compute the intermediate signal $w_t(\tau)$ via pointwise multiplication of $x(\tau)$ and $h(t-\tau)$. Then, integrate over the entire real line to find $y(t)$.

Example 1. *Calculate $y(t) = x(t) * h(t)$ when $x(t) = h(t) = u(t) - u(t-1)$.*

Example 2. *Consider the CT system described by the LCCDE*

$$2\frac{dy(t)}{dt} + y(t) = x(t)$$

Assuming the condition of initial rest, calculate the response to the input signal $x(t) = e^t u(t)$.

Since the system is LTI, let us first calculate the unit impulse response $h(t)$ of the system and then use the convolution integral to calculate the response $y(t)$ to the input signal $x(t) = e^t u(t)$. To calculate the unit impulse response, let us assume that $x(t) = \delta(t)$. Since $x(t) = 0$ for $t > 0$, the particular solution $y_p(t) = 0$ for all $t \in \mathbb{R}$. The homogeneous solution $y_h(t)$, as calculated previously in Lecture 3, is given by

$$y_h(t) = Ae^{-t/2}$$

Thus, the overall solution

$$y(t) = y_p(t) + y_h(t) = Ae^{-t/2}, \quad \forall t > 0$$

To determine the value of A, we may determine the value of $y(t)$ at $t = 0^+$ from the condition of initial rest as follows. Integrating the LCCDE from 0^- to 0^+, we have

$$2\int_{0^-}^{0^+} \frac{dy(t)}{dt}dt + \int_{0^-}^{0^+} y(t)dt = \int_{0^-}^{0^+} x(t)dt$$

giving

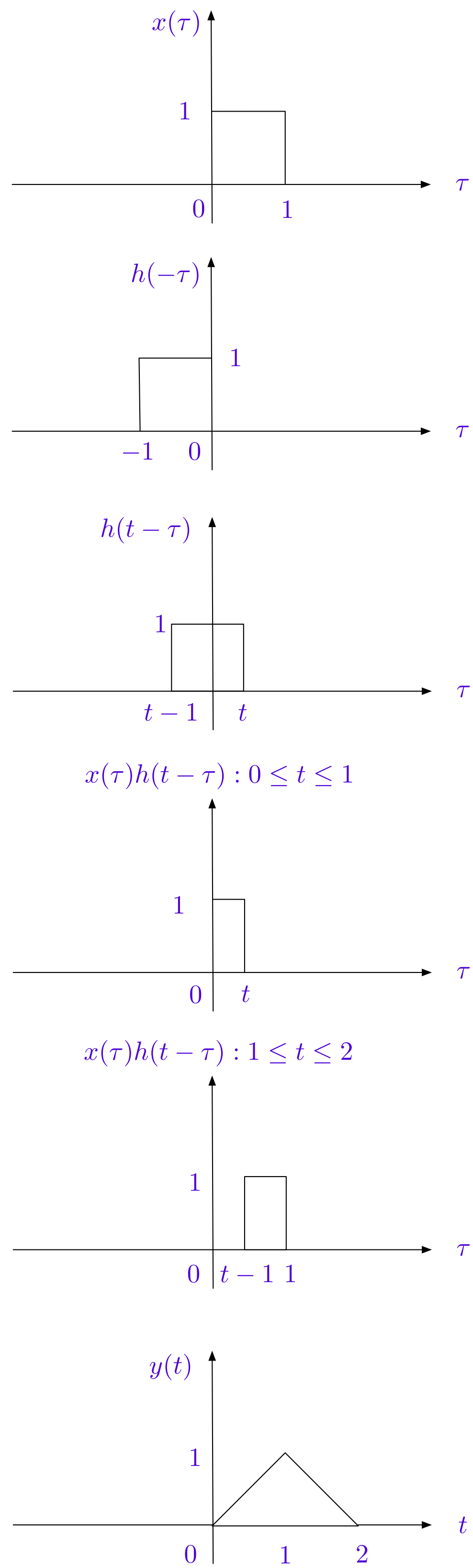
$$2(y(0^+) - y(0^-)) = 1 \quad \Longrightarrow \quad y(0^+) = y(0^-) + \frac{1}{2} = \frac{1}{2}$$

Substituting $y(0^+) = 1/2$ into the overall solution, we can determine that $A = 1/2$, giving the unit impulse response

$$h(t) = \frac{1}{2}e^{-t/2}u(t)$$

Following a similar graphical procedure to DT convolution, we can calculate the output $y(t)$ corresponding to the input $x(t) = e^t u(t)$ as given by

$$y(t) = x(t) * h(t) = \frac{1}{3}(e^t - e^{-t/2})u(t)$$



$x(\tau)$
1
0
1
τ
$h(-\tau)$
1
-1
0
τ
$h(t-\tau)$
1
$t-1$
t
τ
$x(\tau)h(t-\tau) : 0 \leq t \leq 1$
1
0
t
τ
$x(\tau)h(t-\tau) : 1 \leq t \leq 2$
1
0
$t-1$
1
τ
$y(t)$
1
0
1
2
t