# ECEN 314: Signals and Systems

Lecture Notes 7: The Response of LTI Systems to Complex Exponentials

**Reading:**

- Current: SSOW 3.1-3.2
- Next: SSOW 3.3-3.4

## 1 Motivating Example

Consider the LTI system $y[n] = x[n] + x[n-2]$ with the input $\cos(\omega t + \phi)$. What can we say about the output in this case? Using the identity $\cos(a+b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$, one finds that

$$\begin{aligned} y[n] &= \cos(\omega n + \phi) + \cos(\omega(n-2) + \phi) \\ &= 2\cos(\omega)\cos(\omega n + (\phi - \omega)). \end{aligned}$$

The result is that we get an output waveform of the same frequency, but with a phase shift and an amplitude scaling. Therefore, the input-output mapping for sinusoidal inputs only consists of a scale factor and phase shift. It is not too hard to generalize this and show that the same thing happens for arbitrary LTI systems! In the next section, we will use the power of complex numbers to derive stronger results with even simpler arguments.

## 2 Discrete-time signals

Consider the DT complex exponential input $x[n] = z^n$, where $z = re^{j\Omega}$ is an arbitrary complex number. The exponential growth rate of this signal is determined by $r$ and the oscillation period is given by $N = \text{lcm}(1, 2\pi/\Omega)$ (i.e., the smallest multiple of $2\pi/\Omega$ that is also integer). For a DT LTI system with impulse response $h[n]$, the associated output is given by the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \underbrace{\sum_{k=-\infty}^{\infty} h[k]z^{-k}}_{H(z)}.$$

This implies that the output signal is equal to the input signal multiplied by the complex constant $H(z)$, which depends explicitly on $z$ and implicitly on the system impulse response $h[n]$. The function $H(z)$ is called the *transfer function* of the LTI system.

In linear algebra, an *eigenvector* of a linear transform is an input whose output equals the input multiplied by a constant. For example, a vector $v$ is an eigenvector of a matrix $A$ if $Av = \lambda v$ for some scalar $\lambda$. Since we view a signal $x[n]$ as a function of $n$, we say that the complex exponential $x[n] = z^n$ is an *eigenfunction* of a DT LTI system because this input produces the input multiplied the eigenvalue $H(z)$ for any DT LTI system.

## 2.1 Basic DT example

As an example, consider the LTI system $x[n] \to y[n] = x[n] + 0.5x[n-1]$ with impulse response $h[n] = \delta[n] + 0.5\delta[n-1]$. For this system, the transfer function is given by

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k} = \sum_{k=-\infty}^{\infty} (\delta[n] + 0.5\delta[n-1])z^{-k} = 1 + 0.5z^{-1}.$$

Therefore, the above analysis shows that the input $x[n] = 0.25^n$ will produce the output

$$y[n] = H(0.25)x[n] = (1 + 0.5/0.25)x[n] = 3(0.25)^n.$$

In this simple example, this input-output relationship might also be observed directly

$$y[n] = x[n] + 0.5x[n-1] = (0.25)^n + 0.5(0.25)^{n-1} = (1 + 0.5/0.25)(0.25)^n = 3(0.25)^n.$$

For complicated problems, however, we will find that the transfer function approach is very powerful.

## 2.2 Series DT example

Now, consider the LTI system, with impulse response $h[n]$, defined by the series connnection of two LTI systems, with impulse responses $h_1[n]$ and $h_2[n]$. Let the input signal be $x[n] = z^n$, the output of the first system be $w[n]$, and the overall output be $y[n]$. This implies that $w[n] = H_1(z)x[n]$, where the transfer function of the first system is

$$H_1(z) = \sum_{k=-\infty}^{\infty} h_1[k]z^{-k}.$$

Likewise, $y[n] = H_2(z)w[n]$, where the transfer function of the second system is

$$H_2(z) = \sum_{k=-\infty}^{\infty} h_2[k]z^{-k}.$$

Therefore, the output is given by $y[n] = H_1(z)H_2(z)z^n$ and we see the transfer function $H(z) = H_1(z)H_2(z)$ of the overall system is given simply by the product of the individual transfer functions.

# 3 Continuous-time signals

Consider the CT complex exponential input $x(t) = e^{st}$, where $s = \sigma + j\omega$ is an arbitrary complex number. The exponential growth rate of this signal is determined by $\sigma$ and the oscillation period is given by $2\pi/\omega$. For a CT LTI system with impulse response $h(t)$, the associated output is given by the convolution sum

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau}_{H(s)}.$$

This implies that the output signal is equal to the input signal multiplied by the complex constant $H(s)$, which depends explicitly on $s$ and implicitly on the system impulse response $h(t)$.

Since we view a signal $x(t)$ as a function of $t$, we say that the complex exponential $x(t) = e^{st}$ is an *eigenfunction* of a CT LTI system because this input produces the input multiplied the eigenvalue $H(s)$ for any CT LTI system.

## 3.1 Basic CT example

As an example, consider the LTI system $x(t) \to y(t) = x(t) - x(t - 1/2)$ with impulse response $h(t) = \delta(t) + \delta(t - 1/2)$. For this system, the transfer function is given by

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = \int_{-\infty}^{\infty} (\delta(t) + \delta(t - 1/2)) = 1 + e^{-s/2}.$$

Therefore, the above analysis shows that the input $x(t) = e^{j\pi t}$ will produce the output

$$y(t) = H(j\pi)x(t) = (1 + e^{-j\pi/2})x(t) = (1 - j)e^{j\pi t}.$$