ECEN 314: Signals and Systems

Lecture Notes 8: CT Fourier Series

Reading:

- Current: SSOW 3.3-3.4
- Next: SSOW 3.5

1 Inner Products

Recall that the inner product (or dot product) between two m-dimensional real column vectors $\underline{u} = (u_1, \ldots, u_m)$ and $\underline{v} = (v_1, \ldots, v_m)$ is given by

$$\underline{u} \cdot \underline{v} = \underline{v}^T \underline{u} = \sum_{k=1}^{m} u_k v_k.$$

For complex vectors, one uses the Hermitian transpose $\underline{v}^H \triangleq (\underline{v}^T)^*$ instead and this gives

$$\langle \underline{u}, \underline{v} \rangle \triangleq \underline{v}^H \underline{u} = \sum_{k=1}^{m} u_k v_k^*,$$

where $*$ denotes complex conjugation. Two vectors $\underline{u}, \underline{v}$ are *orthogonal* if $\langle \underline{u}, \underline{v} \rangle = 0$ and $\underline{u}$ is normalized if $\langle \underline{u}, \underline{u} \rangle = 1$. A set of vectors, $\underline{u}_1, \ldots, \underline{u}_M$ is called orthogonal if they are pairwise orthogonal and have the same length (i.e., $\langle \underline{u}_k, \underline{u}_n \rangle = \delta[k-n] \langle \underline{u}_n, \underline{u}_n \rangle$).

The main benefit of an orthogonal set of vectors is that one can easily identify coefficients in a linear combination. For example, let $\underline{u}_1, \ldots, \underline{u}_M$ be an orthogonal set and consider the linear combination

$$\underline{v} = \sum_{k=1}^{M} a_k \underline{u}_k.$$

Then, we can compute

$$\langle \underline{v}, \underline{u}_n \rangle = \underline{u}_n^H \sum_{k=1}^{M} a_k \underline{u}_k = \sum_{k=1}^{M} a_k \left(\underline{u}_n^H \underline{u}_k \right) = \sum_{k=1}^{M} a_k \delta[k-n] \langle \underline{u}_n, \underline{u}_n \rangle = a_n \langle \underline{u}_n, \underline{u}_n \rangle.$$

So, taking the inner product with the the n-th vector in the set recovers the coefficient of the n-th vector in the sum multiplied by a constant.

2 Signal Decomposition

For LTI systems, our introduction of the impulse response was based on the observation that the input signal $x[n]$ could be decomposed into linear combinations of basic signals: $\psi_k[n] = \delta[n-k]$ for all integer k. This is an orthonormal basis for signals is known as the *time domain* and we can write and signal $x[n]$ as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\psi_k[n].$$

Given the simple input-output relationship we now have for complex exponentials, a decomposition of arbitrary signals into linear combinations of complex exponentials would be useful. In this lecture, we will see that this is indeed possible using Fourier analysis. In particular, there is another orthonormal basis for signals known as the *frequency domain*. We start by focusing on periodic CT signals with period T.

3 Periodic CT signals

A CT signal $x(t)$ is said to be *periodic* if there exists $T > 0$ such that

$$x(t+T) = x(t), \quad \forall t \in \mathbb{R}.$$

The smallest such T is called the *fundamental period* and is usually denoted as T_0. The *fundamental frequency* is defined as

$$\omega_0 \triangleq \frac{2\pi}{T_0}$$

Recall that, if $x(t)$ is periodic with period T, then $x(t+kT) = x(t)$ for any positive *integer* k.

Examples:

- The real sinusoid $x(t) = \cos(\omega_0 t + \theta)$ is periodic with the fundamental frequency $|\omega_0|$ and the fundamental period $T = 2\pi/|\omega_0|$.

- The complex sinusoid $x(t) = e^{j\omega_0 t}$ is periodic with the fundamental frequency $|\omega_0|$ and the fundamental period $T = 2\pi/|\omega_0|$.

Combining Periodic CT Signals:

- Combining periodic CT signals may not always result in periodic signals. In particular, if any of the ratios of the fundamental periods of the individual signals is irrational, then the overall signal may not be periodic. If those ratios are all rational, then a period is given by the lowest common multiple (LCM) of the periods of the individual signals.

- It is possible that the fundamental period of the overall signal is smaller than the LCM of the fundamental periods of the individual signals.

Examples:

- $x(t) = \cos\left(\frac{3\pi}{2}t\right) + 3\sin\left(\frac{\pi}{3}t\right)$
- $x(t) = \cos\left(\frac{3\pi}{2}t\right) + 3\sin\left(t\right)$
- $x(t) = \cos(2\pi t) \cdot \cos(2\pi t)$

4 CT Fourier series

In the 18th century, Jean Baptiste Joseph Fourier discovered that a wide class of periodic signals $x(t)$ can be written as linear combinations of the basic signals $\phi_k(t) = e^{jk\omega_0 t}$ where ω_0 is the fundamental frequency of $x(t)$, i.e.,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \tag{1}$$

Frequency terminology:

- $\{a_k\}$: Fourier series coefficients
- a_0: DC component
- $a_{\pm 1}$: fundamental component (first harmonic)
- $a_{\pm 2}$: second harmonic
- $a_{\pm k}$: kth harmonic.

This important observation led to two natural questions:

1. (Existence of Fourier series representation) Which periodic signals can indeed be represented as linear combinations of a complex exponential function and its harmonics?
2. If the Fourier series representation indeed exists, how can we determine the Fourier series coefficients?

First, we will answer the second question for any signal of the form (1). After that, we will return to question #1 and identify sets of functions for which the Fourier series representation equals the original function.

5 Finding Fourier series coefficients

5.1 Simple example

For simple periodic signals consisting of a few sinusoidal terms, this can be done by considering the following "inverse" Euler's identity:

$$\begin{aligned}\cos\theta &= \frac{1}{2}(e^{j\theta}+e^{-j\theta})\\ \sin\theta &= \frac{1}{j2}(e^{j\theta}-e^{-j\theta})\end{aligned}$$

Example: Consider the signal

$$x(t) = 1 + 2\cos(4\pi t+\pi/6) - 5\sin(8\pi t)$$

with the fundamental frequency $\omega_0 = 4\pi$. By the inverse Euler's identity,

$$\begin{aligned}x(t) &= 1 + 2\cos(4\pi t+\pi/6) - 5\sin(8\pi t)\\ &= 1 + \left[e^{j(4\pi t+\pi/6)} + e^{-j(4\pi t+\pi/6)}\right] - \frac{5}{j2}\left[e^{j8\pi t} - e^{-j8\pi t}\right]\\ &= 1 + \left[e^{j\pi/6}e^{j\omega_0 t} + e^{-j\pi/6}e^{-j\omega_0 t}\right] + j\frac{5}{2}\left[e^{j2\omega_0 t} - e^{-j2\omega_0 t}\right]\end{aligned}$$

Conclusion:

$$a_k = \begin{cases} 1, & k=0\\ e^{j\pi/6}, & k=1\\ e^{-j\pi/6}, & k=-1\\ j\frac{5}{2}, & k=2\\ -j\frac{5}{2}, & k=-2\\ 0, & \text{else}\end{cases}$$

5.2 General case

To answer question #2 in general, we let $\omega_0 = 2\pi/T$ and assume

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

If we multiply both sides of this equation by $e^{-jk\omega_0 t}$ and integrate over one period, then

$$\begin{aligned}\int_0^T x(t)e^{-jn\omega_0 t}dt &= \int_0^T \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right) e^{-jn\omega_0 t}dt\\ &= \sum_{k=-\infty}^{\infty} a_k \left(\int_0^T e^{j(k-n)\omega_0 t}dt\right)\\ &= \sum_{k=-\infty}^{\infty} a_k \delta[n-k]T = a_n T.\end{aligned}$$

The above integral was evaluated by observing there are two important cases: $k = n$ and $k \neq n$. If $k = n$, then we have

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T 1 dt = T.$$

If $k \neq n$, then we substitute $\omega_0 = 2\pi/T$ and write

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T e^{j(k-n)(2\pi/T)t} dt = \frac{1}{j(k-n)(2\pi/T)} \left(e^{j(k-n)2\pi} - 1 \right) = 0.$$

This is very similar to the case of orthogonal vectors. To make the connection precise, we define the inner product,

$$\langle x(t), y(t) \rangle \triangleq \int_0^T x(t) y^*(t) dt,$$

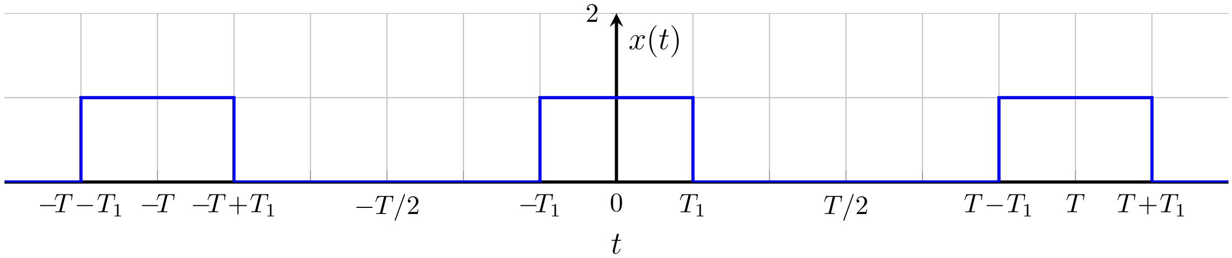
between functions mapping $[0, T]$ to the complex numbers. Under this inner product, the family of functions $\phi_k(t) = e^{jk\omega_0 t}$ is an orthogonal set and, from above, we have

$$\langle \phi_k(t), \phi_n(t) \rangle = \int_0^T e^{j(k-n)\omega_0 t} dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

CT Fourier series Analysis and Synthesis:

$$\begin{aligned} x(t) = & \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} & \text{(Synthesis equation)} \\ a_k = & \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt & \text{(Analysis equation)} \end{aligned}$$

5.3 Example: Periodic square wave



Consider the Fourier series analysis equation for the periodic square wave (with period T) where one period is defined by

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < \frac{T}{2}. \end{cases}$$

Since the function is periodic, we can compute the integral over any full period and the choice $-T/2$ to $T/2$ is quite convenient in this case. For $k = 0$, we get

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt = \frac{1}{T} \int_{-T_1}^{T_1} dt = \frac{2T_1}{T}.$$

The DC component is just the average and we also call this the *duty cycle* $d \triangleq 2T_1/T$.

For $k \neq 0$, we get

$$\begin{aligned} a_k &= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{T} \int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt \\ &= -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t} \Big|_{-T_1}^{T_1} \\ &= \frac{\sin(k\omega_0 T_1)}{k\pi} \\ &= \frac{\sin(k 2\pi T_1/T)}{k\pi} \end{aligned}$$

where we used the fact that $\omega_0 = 2\pi/T$. To summarize, the Fourier series coefficients of a periodic square wave of duty cycle $d = 2T_1/T$ are given by

$$a_k = \begin{cases} d, & k = 0 \\ \frac{\sin(k\pi d)}{k\pi}, & k \neq 0. \end{cases}$$

From this, it is easy to verify that $a_k = a_{-k}$. Plugging this into the synthesis formula shows that

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\ &= a_0 + \sum_{k=1}^{\infty} \left(a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} \right) \\ &= a_0 + \sum_{k=1}^{\infty} a_k \left(e^{jk\omega_0 t} + e^{-jk\omega_0 t} \right) \\ &= a_0 + \sum_{k=1}^{\infty} a_k \left(2\cos(k\omega_0 t) \right) \\ &= d + 2\sum_{k=1}^{\infty} \frac{\sin(k\pi d)}{k\pi} \cos(k\omega_0 t). \end{aligned}$$

Moreover, the standard square wave with duty cycle $d = \frac{1}{2}$ leads to the simplification

$$\sin(k\pi d) = \sin(k\pi/2) = \begin{cases} 0 & \text{if } k \text{ even} \\ (-1)^{(k-1)/2} & \text{if } k \text{ odd.} \end{cases}.$$

Substituting this into the above formula gives the sum of odd harmonics given by

$$x(t) = \frac{1}{2} + 2\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)\pi} \cos((2m-1)\omega_0 t).$$

6 Convergence of CT Fourier series

To understand convergence, we consider the question:

$$\text{“If we choose } a_k = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} dt, \text{ then when do we have } x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\text{ ?”}$$

For infinite sums of functions, the question of convergence (i.e., what we mean by equality in the above expression) is somewhat subtle. A useful definition for engineering is that, in the limit, there is no *energy* in the difference signal

$$e_N(t) \triangleq x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}.$$

In particular, this requires that

$$\lim_{N\to\infty} \int_0^T |e_N(t)|^2 dt = \lim_{N\to\infty} \int_0^T \left| x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \right|^2 dt = 0.$$

It can be shown that this occurs whenever the periodic signal $x(t)$ has finite energy over a fundamental period, i.e.,

$$\int_0^T |x(t)|^2 dt < \infty.$$

However, this does *not* mean the signal $x(t)$ and its Fourier series representation are equal at every value of t.

For real-world signals that satisfy the following Dirichlet conditions, the Fourier series representation will indeed equal $x(t)$ at points where $x(t)$ is continuous. At points of discontinuity, it equals the midpoint

$$\frac{x(t^-) + x(t^+)}{2}.$$

The Dirichlet conditions are:

1. $x(t)$ is absolutely integrable over a fundamental period, i.e.,

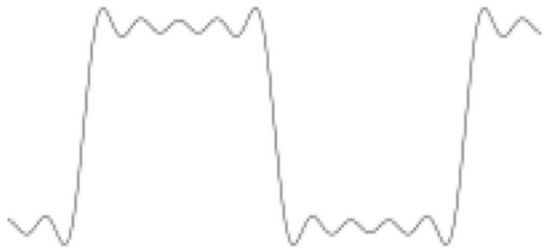
$$\int_0^T |x(t)| dt < \infty$$

2. $x(t)$ only has a finite number of maxima and minima in each fundamental period
3. $x(t)$ only has a finite number of finite jump discontinuities in each fundamental period

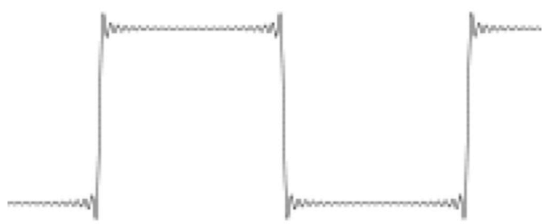
7 Gibbs phenomenon

For signals that satisfy the Dirichlet conditions (i.e., the periodic square wave), the Fourier series representation converges to the original signal at points where the signal is continuous and to the midpoint at points of discontinuity. This convergence exhibits the following interesting phenomenon known as the Gibbs phenomenon (named after the American physicist Josiah Willard Gibbs).

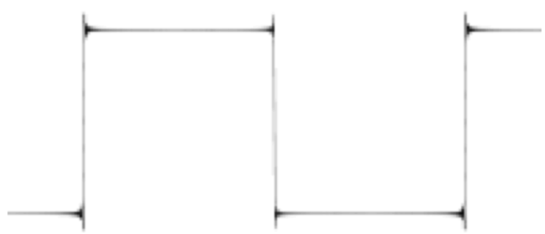
As we can see, as we attempt to reconstruct the signal from its Fourier series coefficients, the more coefficients we use, the more the signal begins to resemble the original. However, around the discontinuities, we observe rippling that does not seem to subside. As we consider even more coefficients, we notice that the ripples narrow, but do not shorten. As we approach an infinite number of coefficients, this rippling still does not go away (never dropping below 9% of the wave height). On the other hand, the area inside them tends to zero, meaning that the energy of this ripple goes to zero. This means that their width is approaching zero and we can assert that the reconstruction is exactly the original except at the points of discontinuity.



Fourier series representation of
periodic square wave using 5 harmonics



Fourier series representation of
periodic square wave using 25 harmonics



Fourier series representation of
periodic square wave using 125 harmonics

8 Integrating over one period of a periodic function

Let $f(t)$ be a periodic function with period T that is integrable over one period. In this case, the integral over any interval of length T gives the same answer and we use the notation

$$\int_T f(t)\,dt$$

to emphasize that the answer does not depend on the interval chosen.

Proof. For any $0 \leq t_0 \leq T$ and integer k, we have

$$\begin{aligned}
\int_{kT+t_0}^{(k+1)T+t_0} f(t)dt &= \int_{t_0}^{t_0+T} f(t+kT)dt \\
&= \int_{t_0}^{t_0+T} f(t)dt \\
&= \int_{t_0}^{T} f(t)dt + \int_{T}^{T+t_0} f(t)dt \\
&= \int_{t_0}^{T} f(t)dt + \int_{0}^{t_0} f(t+T)dt \\
&= \int_{t_0}^{T} f(t)dt + \int_{0}^{t_0} f(t)dt \\
&= \int_{0}^{T} f(t)dt.
\end{aligned}$$

$\square$