ECEN 314: Signals and Systems

Lecture Notes 9: Properties of CTFS

Reading:

- Current: SSOW 3.5
- Next: SSOW 3.6

Property 1 (Linearity)**.** *Let $x(t)$ and $y(t)$ be two periodic CT signals with the same fundamental period T. Suppose that $x(t) \overset{FS}{\longleftrightarrow} a_k$ and $y(t) \overset{FS}{\longleftrightarrow} b_k$. Then*

$$\alpha x(t) + \beta y(t) \overset{FS}{\longleftrightarrow} \alpha a_k + \beta b_k$$

Proof. Let $\alpha x(t) + \beta y(t) \overset{FS}{\longleftrightarrow} c_k$. By the analysis equation,

$$\begin{aligned} c_k &= \frac{1}{T}\int_0^T (\alpha x(t) + \beta y(t)) e^{-jk\omega_0 t} dt \\ &= \frac{\alpha}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt + \frac{\beta}{T}\int_0^T y(t)e^{-jk\omega_0 t}dt \\ &= \alpha a_k + \beta b_k \end{aligned}$$

□

Property 2 (Time Reversal)**.** *Suppose that $x(t) \overset{FS}{\longleftrightarrow} a_k$. Then*

$$x(-t) \overset{FS}{\longleftrightarrow} a_{-k}$$

Proof. Let $x(-t) \overset{FS}{\longleftrightarrow} b_k$. By the analysis equation,

$$\begin{aligned} b_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(-t)e^{-jk\omega_0 t}dt \\ &= -\frac{1}{T}\int_{T/2}^{-T/2} x(t)e^{jk\omega_0 t}dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j(-k)\omega_0 t}dt \\ &= a_{-k} \end{aligned}$$

□

Property 3 (Even/Odd Symmaetry)**.** *Suppose that* $x(t) \overset{FS}{\longleftrightarrow} a_k$*. If* $x(t)$ *is even, then* a_k *is also even, i.e.,* $a_{-k} = a_k$*. If* $x(t)$ *is odd, then* a_k *is also odd, i.e.,* $a_{-k} = -a_k$*.*

Proof. When $x(t)$ is even, we have $x(-t) = x(t)$. By the time-flip property,

$$a_{-k} = a_k$$

When $x(t)$ is odd, we have $x(-t) = -x(t)$. By the time-flip and linearity properties,

$$a_{-k} = -a_k$$

$\square$

Definition 1 (Conjugate Symmetry)**.** *A CT signal* $x(t)$ *has conjugate symmetry if* $x(-t) = x^*(t)$*. Likewise, a DT signal* $x[n]$ *(or sequence* a_k*) has conjugate symmetry if* $x[-n] = x^*[n]$ *(or* $a_{-k} = a_k^*$*).*

Property 4 (Conjugation)**.** *Suppose that* $x(t) \overset{FS}{\longleftrightarrow} a_k$*. Then*

$$x^*(t) \overset{FS}{\longleftrightarrow} a_{-k}^*$$

Proof. Let $x^*(t) \overset{FS}{\longleftrightarrow} b_k$. By the analysis equation,

$$\begin{aligned}
b_k &= \frac{1}{T}\int_0^T x^*(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{T}\int_0^T x^*(t)\left(e^{jk\omega_0 t}\right)^* dt \\
&= \frac{1}{T}\int_0^T \left(x(t)e^{jk\omega_0 t}\right)^* dt \\
&= \left(\frac{1}{T}\int_0^T x(t)e^{jk\omega_0 t}dt\right)^* \\
&= \left(\frac{1}{T}\int_0^T x(t)e^{-j(-k)\omega_0 t}dt\right)^* \\
&= a_{-k}^*
\end{aligned}$$

$\square$

Property 5 (Conjugate Symmetry)**.** *Suppose that* $x(t) \overset{FS}{\longleftrightarrow} a_k$ *and* $x(t)$ *is real. Then*

$$a_{-k} = a_k^*$$

Proof. If $x(t)$ is real, then $x^*(t) = x(t)$. By the conjugate property,

$$a_k = a_{-k}^*$$

By the simple change of variable $k \to -k$, we have

$$a_{-k} = a_k^*$$

$\square$

Remark 1. *Note that*

$$\begin{aligned} a_k &= Re\{a_k\} + jIm\{a_k\} \\ &= |a_k| e^{j\angle a_k} \end{aligned}$$

so we have

$$\begin{aligned} a_{-k} &= Re\{a_{-k}\} + jIm\{a_{-k}\} \\ &= |a_{-k}| e^{j\angle a_{-k}} \end{aligned}$$

and

$$\begin{aligned} a_k^* &= Re\{a_k\} - jIm\{a_k\} \\ &= |a_k| e^{-j\angle a_k} \end{aligned}$$

Thus, if $x(t)$ is real, then $Re\{a_k\}$ is even, $Im\{a_k\}$ is odd, $|a_k|$ is even, and $\angle a_k$ is odd.

Remark 2. *If $x(t)$ is real, by the synthesis equation*

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\ &= a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} \right] \\ &= a_0 + \sum_{k=1}^{\infty} \left[|a_k| e^{j\angle a_k} e^{jk\omega_0 t} + |a_{-k}| e^{j\angle a_{-k}} e^{-jk\omega_0 t} \right] \\ &= a_0 + \sum_{k=1}^{\infty} \left[|a_k| e^{j\angle a_k} e^{jk\omega_0 t} + |a_k| e^{-j\angle a_k} e^{-jk\omega_0 t} \right] \\ &= a_0 + \sum_{k=1}^{\infty} |a_k| \left[e^{j(k\omega_0 t + \angle a_k)} + e^{-j(k\omega_0 t + \angle a_k)} \right] \\ &= a_0 + 2 \sum_{k=1}^{\infty} |a_k| \cos(k\omega_0 t + \angle a_k). \end{aligned}$$

This is the form of CTFS for real signals that is known to many people.

Remark 3. *Combining Properties 2 and 4, we see that if $x(t)$ is real and even, then*

$$a_{-k} = a_k^* = a_k$$

i.e., a_k is real and even; and if $x(t)$ is real and odd, then

$$a_{-k} = a_k^* = -a_k$$

i.e., a_k is pure imaginary and odd.

Property 6 (Time Shift)**.** *Suppose that* $x(t) \overset{FS}{\longleftrightarrow} a_k$. *Then*

$$x(t-t_0) \overset{FS}{\longleftrightarrow} a_k e^{-jk\omega_0 t_0}.$$

In particular, if $t_0 = T/2$, *then*

$$x(t-T/2) \overset{FS}{\longleftrightarrow} (-1)^k a_k$$

Proof. Let $x(t-t_0) \overset{FS}{\longleftrightarrow} b_k$. By the analysis equation,

$$\begin{aligned} b_k &= \frac{1}{T}\int_0^T x(t-t_0)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 (t+t_0)}dt \\ &= e^{-jk\omega_0 t_0}\frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt \\ &= e^{-jk\omega_0 t_0}a_k \end{aligned}$$

When $t_0 = T/2$, we have $\omega_0 t_0 = \pi$. Note that

$$e^{-jk\pi} = (-1)^k.$$

We thus have $b_k = (-1)^k a_k$. □

Property 7 (Multiplication)**.** *Let* $x(t)$ *and* $y(t)$ *be two periodic CT signals with the same fundamental period* T. *Suppose that* $x(t) \overset{FS}{\longleftrightarrow} a_k$ *and* $y(t) \overset{FS}{\longleftrightarrow} b_k$. *Then*

$$x(t)y(t) \overset{FS}{\longleftrightarrow} a_k * b_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

Proof. By the synthesis equation,

$$x(t) = \sum_{l=-\infty}^{\infty} a_l e^{jl\omega_0 t}$$

and

$$y(t) = \sum_{m=-\infty}^{\infty} b_m e^{jm\omega_0 t}.$$

We thus have

$$\begin{aligned} x(t)y(t) &= \left(\sum_{l=-\infty}^{\infty} a_l e^{jl\omega_0 t}\right)\cdot\left(\sum_{m=-\infty}^{\infty} b_m e^{jm\omega_0 t}\right) \\ &= \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_l b_m e^{j(l+m)\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{\infty} a_l b_{k-l}\right)e^{jk\omega_0 t} \end{aligned}$$

where the last equality follows from the change of variable $(l, m) \to (l, k)$ where $k = l + m$. Let $x(t)y(t) \overset{FS}{\longleftrightarrow} c_k$ and we have

$$x(t)y(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Comparing the previous two equations, we conclude that

$$c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} = a_k * b_k.$$

$\square$

Property 8 (Parseval's Relation)**.** *Let $x(t)$ be a periodic CT signal with fundamental period T. Suppose that $x(t) \overset{FS}{\longleftrightarrow} a_k$. Then*

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

Remark 4. *Note that $\frac{1}{T}\int_0^T |x(t)|^2 dt$ represents the average power of $x(t)$ measured in the time domain, and $|a_k|^2$ represents the power of the kth harmonic of $x(t)$. So the Parseval's relation basically states the power of $x(t)$ is the same whether it is measured in the time or frequency domain.*

Proof. We calculate directly

$$\begin{aligned}
\frac{1}{T}\int_0^T |x(t)|^2 dt &= \frac{1}{T}\int_0^T \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right)\left(\sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}\right)^* dt \\
&= \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_k a_n^* \left(\frac{1}{T}\int_0^T e^{j(k-n)\omega_0 t} dt\right) \\
&= \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_k a_n^* \delta[n-k] \\
&= \sum_{k=-\infty}^{\infty} |a_k|^2.
\end{aligned}$$

$\square$

Alternate Proof. Let $x(t) \overset{FS}{\longleftrightarrow} a_k$, $x^*(t) \overset{FS}{\longleftrightarrow} b_k$, and $|x(t)|^2 = x(t)x^*(t) \overset{FS}{\longleftrightarrow} c_k$. Consider the analysis equation with $k = 0$, and we have

$$c_0 = \frac{1}{T}\int_0^T z(t) dt = \frac{1}{T}\int_0^T |x(t)|^2 dt.$$

By the conjugate and multiplication properties, $b_k = a_{-k}^*$ and $c_k = a_k * b_k$. We thus have

$$c_0 = \sum_{l=-\infty}^{\infty} a_l b_{-l} = \sum_{l=-\infty}^{\infty} a_l a_l^* = \sum_{l=-\infty}^{\infty} |a_l|^2.$$

We thus conclude that

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{l=-\infty}^{\infty} |a_l|^2.$$

□

Let $x(t)$ and $y(t)$ be two periodic CT signals with the same fundamental period T.

Definition 2 (Periodic Convolution). *The periodic convolution is defined by*

$$z(t) = x(t) \circledast y(t) \triangleq \frac{1}{T}\int_0^T x(\tau)y(t-\tau)d\tau = \frac{1}{T}\int_0^T x(t-\tau)y(\tau)d\tau.$$

It is easy to verify that $z(t)$ is also periodic with a fundamental period T.

Property 9 (Periodic Convolution). *Let $x(t)$ and $y(t)$ be two periodic CT signals with the same fundamental period T, and let $z(t)$ be the periodic convolution of $x(t)$ and $y(t)$. Suppose that $x(t) \overset{FS}{\longleftrightarrow} a_k$ and $y(t) \overset{FS}{\longleftrightarrow} b_k$. Then*

$$z(t) \overset{FS}{\longleftrightarrow} a_k b_k.$$

Proof. Let $z(t) \overset{FS}{\longleftrightarrow} c_k$. By the analysis equation,

$$\begin{aligned} c_k &= \frac{1}{T}\int_0^T z(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T^2}\int_0^T \left(\int_0^T x(\tau)y(t-\tau)d\tau\right) e^{-jk\omega_0 t}dt. \end{aligned}$$

Consider the simple change of variable $t - \tau \to t$, and we have

$$\begin{aligned} c_k &= \frac{1}{T^2}\int_0^T \left(\int_0^T x(\tau)y(t)d\tau\right) e^{-jk\omega_0 (t+\tau)}dt \\ &= \left(\frac{1}{T}\int_0^T x(\tau)e^{-jk\omega_0 \tau}d\tau\right)\left(\frac{1}{T}\int_0^T y(t)e^{-jk\omega_0 t}\right) dt \\ &= a_k b_k. \end{aligned}$$

□

Property 10 (Differentiation)**.** *Suppose that* $x(t) \overset{FS}{\longleftrightarrow} a_k$*. Then*

$$\frac{d}{dt}x(t) \overset{FS}{\longleftrightarrow} jk\omega_0 a_k$$

In particular, the DC component of $dx(t)/dt$ *is always equal to zero.*

Proof. By the synthesis equation,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

Taking derivative on both sides, we have

$$\begin{aligned} \frac{d}{dt}x(t) &= \sum_{k=-\infty}^{\infty} a_k \frac{d}{dt} e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} (jk\omega_0 a_k) e^{jk\omega_0 t}. \end{aligned}$$

Let $dx(t)/dt \overset{FS}{\longleftrightarrow} b_k$. Then,

$$\frac{d}{dt}x(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}.$$

Comparing the previous two equations, we conclude that

$$b_k = jk\omega_0 a_k.$$

□

Property 11 (Time Scaling)**.** *Suppose that* $x(t) \overset{FS}{\longleftrightarrow} a_k$ *is periodic with period* T *and let* $y(t) = x(mt)$ *for some positive integer* m*. Then,* $y(t) \overset{FS}{\longleftrightarrow} b_k$ *(assuming* T *is unchanged) implies that*

$$b_k = \begin{cases} a_{k/m} & \text{if } m \text{ divides } k \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We proceed directly and calculate

$$
\begin{aligned}
b_k &= \frac{1}{T}\int_0^T x(mt)e^{-jk\omega_0 t}dt \\
&= \frac{1}{T}\int_0^{mT} x(s)e^{-jk\omega_0 s/m}\frac{1}{m}ds \\
&= \frac{1}{mT}\sum_{i=0}^{m-1}\int_{iT}^{(i+1)T} x(s)e^{-jk\omega_0 s/m}ds \\
&= \frac{1}{mT}\sum_{i=0}^{m-1}\int_0^T x(s+iT)e^{-jk(2\pi/T)(s+iT)/m}ds \\
&= \frac{1}{mT}\left(\int_0^T x(s)e^{-jk\omega_0 s/m}ds\right)\sum_{i=0}^{m-1} e^{-jk(2\pi i/m)} \\
&= \begin{cases} \frac{1}{T}\int_0^T x(s)e^{-jk\omega_0 s/m}ds = a_{k/m} & \text{if } m \text{ divides } k \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}
$$

because $\sum_{i=0}^{m-1} e^{-jk(2\pi i/m)} = \frac{1-e^{-jk(2\pi)}}{1-e^{-jk(2\pi/m)}} = 0$ unless m divides k. □

Example: Determine the FS coefficients of the periodic CT signal $x(t) = \cos^2(\pi t)$.
Answer: Method 1: Using the double-angle formula

$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

we can rewrite $x(t)$ as

$$x(t) = \frac{1}{2} + \frac{1}{2}\cos(2\pi t).$$

Then, it is clear that the fundamental frequency of $x(t)$ is given by $\Omega_0 = 2\pi$ (and hence the fundamental period $T_0 = 1$). Using Euler's identity, we can further rewrite $x(t)$ as

$$x(t) = \frac{1}{2} + \frac{1}{4}e^{j2\pi t} + \frac{1}{4}e^{-j2\pi t} = \frac{1}{2} + \frac{1}{4}e^{j\omega_0 t} + \frac{1}{4}e^{-j\omega_0 t}.$$

We thus conclude that

$$a_k = \begin{cases} 1/2, & k = 0 \\ 1/4, & k = \pm 1 \\ 0, & \text{else.} \end{cases}$$

Method 2: Note that $x(t) = \cos(\pi t)\cdot\cos(\pi t)$. Let $y(t) = \cos(\pi t)$ and let a_k and b_k be the FS coefficients for $x(t)$ and $y(t)$, respectively. Then, by the multiplication property,

$$a_k = b_k * b_k.$$

Note that the fundamental frequency of $y(t)$ is given by $\Omega_0 = \pi$ (and hence the fundamental period $T_0 = 2$). Rewrite $y(t)$ as

$$y(t) = \frac{1}{2}e^{j\pi t} + \frac{1}{2}e^{-j\pi t} = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

so the FS coefficients

$$b_k = \begin{cases} 1/2, & k = \pm 1 \\ 0, & \text{else.} \end{cases}$$

Using the unit impulse function, we can write b_k in a more compact way as

$$b_k = \frac{1}{2}\delta[k+1] + \frac{1}{2}\delta[k-1].$$

Thus,

$$\begin{aligned} a_k &= b_k * b_k \\ &= \left(\frac{1}{2}\delta[k+1] + \frac{1}{2}\delta[k-1]\right) * \left(\frac{1}{2}\delta[k+1] + \frac{1}{2}\delta[k-1]\right) \\ &= \frac{1}{4}\delta[k+2] + \frac{1}{2}\delta[k] + \frac{1}{4}\delta[k-2] \end{aligned}$$

or equvalently

$$a_k = \begin{cases} 1/2, & k = 0 \\ 1/4, & k = \pm 2 \\ 0, & \text{else.} \end{cases}$$

We notice that the results derived from these two methods are slightly different. The reason is that the "fundamental" periods that we used are different for the different methods. Note that even though 2 is the fundamental period for $y(t)$, the fundamental period for $x(t) = [y(t)]^2$ is 1. So, the second method actually computes the FS for a periodic waveform consisting of 2 periods of $x(t)$.

Previously, when we've calculated the FS coefficients using the analysis equation, we've always use the fundamental period T_0 and hence the fundamental frequency $\Omega_0 = 2\pi/T_0$. In fact, it is also fine to use *any* period T (not just necessarily the fundamental one), as long as the corresponding "fundamental" frequency is chosen as $2\pi/T$ in both synthesis and analysis equations. The time-scaling property of the Fourier series shows that integrating over N periods simply has the effect of inserting $N-1$ zeros between each of the coefficients defined by the fundamental period.