Decorrelating Discrete-Time WSS Processes

Supplemental Material for Information Theory
Henry D. Pfister

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1 Wide-Sense Stationary Processes in Discrete-Time

1.1 The Discrete-Time Fourier Transform

Definition 1.1. A discrete-time random process $\{X_n\}_{n\in\mathbb{Z}}$ is wide-sense stationary (WSS) if $E\left[X_n\right] = m$ and $E\left[X_n^* X_{n+k}\right] = r_k$ for all $k, n \in \mathbb{Z}$. If it is complex, we require further that it be *proper* (i.e., circularly symmetric), which implies that $E\left[X_n\right] = 0$ and $E\left[X_n X_{n+k}\right] = 0$ for all $k, n \in \mathbb{Z}$.

The following result relates the autocorrelation function and the power spectral density (PSD) in multiple ways. First, it shows that the variances of DFT outputs converge uniformly to the continuous PSD. Second, it implies that one can also estimate the PSD by first estimating the autocorrelation and then taking the discrete-time Fourier transform. Finally, it leads naturally to the result that the DFT asymptotically decorrelates WSS stationary noise.

Theorem 1.2 (Wiener-Khinchin). *If $\{X_n\}_{n\in\mathbb{Z}}$ is WSS with mean 0 and the autocorrelation, $r_k = E\left[X_n X_{n+k}\right]$, satisfies $\sum_{k=0}^{\infty} |r_k| < \infty$, then the power spectral density*

$$S_X(\omega) \triangleq \lim_{N\to\infty} E\left[\left|\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} X_n e^{-\mathrm{i}\omega n}\right|^2\right] = \sum_{n=-\infty}^{\infty} r_n e^{-\mathrm{i}\omega n}$$

is continuous because both limits converge uniformly for all ω.

Proof. We start by defining

$$\hat{X}_N(\omega) \triangleq \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} X_n e^{-\mathrm{i}\omega n}$$

and observing that

$$\begin{aligned}
E\left[\left|\hat{X}_N(\omega)\right|^2\right] &= E\left[\left(\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} X_n e^{-\mathrm{i}\omega n}\right)^* \left(\frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} X_m e^{-\mathrm{i}\omega m}\right)\right] \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} e^{-\mathrm{i}\omega(m-n)} E\left[X_n^* X_m\right] \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} e^{-\mathrm{i}\omega(m-n)} r_{m-n} \\
&= \frac{1}{N}\sum_{k=-N-1}^{N-1} (N-k)\, e^{-\mathrm{i}\omega k} r_k \\
&= \sum_{k=-N-1}^{N-1} \left(1 - \frac{k}{N}\right) e^{-\mathrm{i}\omega k} r_k.
\end{aligned}$$

This quantity converges uniformly, for all ω, as $N \to \infty$ because r_k is absolutely summable and $\left|\left(1 - \frac{k}{N}\right) e^{-\mathtt{i}\omega k} r_k\right| \leq |r_k|$ whenever it appears in the sum. Therefore, we can interchange the limit and the sum to see that

$$\lim_{N\to\infty} \sum_{k=-N-1}^{N-1} \left(1 - \frac{k}{N}\right) e^{-\mathtt{i}\omega k} r_k = \sum_{k=-\infty}^{\infty} e^{-\mathtt{i}\omega k} r_k,$$

where the RHS exists for all ω because r_k is absolutely summable. Since the LHS is a sequence of continuous functions that converges uniformly, it follows that the RHS is also continuous. □

Lemma 1.3. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be a WSS complex process and $X^N = X_0, \ldots, X_{N-1}$. If Y^N is the DFT of X^N, then the real and imaginary parts Y_n have equal variance and are uncorrelated.*

Proof. A complex random variable X is proper (or circularly symmetric) if $E[X] = 0$ and $E\left[X^2\right] = 0$ (don't forget $XX \neq XX^*$). First, we compute the variance of the real and imaginary parts of an arbitrary proper complex random variable with

$$\sigma_r^2 = E\left[\frac{1}{2}(X + X^*)\frac{1}{2}(X + X^*)\right] = \frac{1}{4}\left(E[XX] + E[X^*X] + E[XX^*] + E[X^*X^*]\right) = \frac{1}{2}E[X^*X]$$

$$\sigma_i^2 = E\left[\frac{1}{2\mathtt{i}}(X - X^*)\frac{1}{2\mathtt{i}}(X - X^*)\right] = -\frac{1}{4}\left(E[XX] - E[X^*X] - E[XX^*] + E[X^*X^*]\right) = \frac{1}{2}E[X^*X].$$

Next, we notice the real and imaginary parts are also uncorrelated for an arbitrary proper complex random variable by observing

$$E\left[\frac{1}{2}(X + X^*)\frac{1}{2\mathtt{i}}(X - X^*)\right] = \frac{1}{4\mathtt{i}}\left(E[XX] + E[X^*X] - E[XX^*] - E[X^*X^*]\right) = 0.$$

Finally, we observe that Y^N is a proper complex random vector because

$$\begin{aligned}
E[Y_jY_k] &= E\left[\hat{X}_N\left(\frac{2\pi j}{N}\right)\hat{X}_N\left(\frac{2\pi k}{N}\right)\right] \\
&= E\left[\left(\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} X_n e^{-\mathtt{i}\frac{2\pi j}{N}n}\right)\left(\frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} X_m e^{-\mathtt{i}\frac{2\pi k}{N}m}\right)\right] \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} e^{-\mathtt{i}\frac{2\pi j}{N}n} e^{-\mathtt{i}\frac{2\pi k}{N}m} \underbrace{E[X_nX_m]}_{0} \\
&= 0.
\end{aligned}$$

□

1.2 Decorrelating Transformations

Let $\{X_n\}_{n\in\mathbb{Z}}$ be a WSS process satisfying $\sum_{k=0}^{\infty}|r_k| < \infty$ and let $[K_{X^N}]_{m,n} \triangleq E[X_n^*X_m] = r_{m-n}$ be the autocorrelation matrix of the vector $X^N = X_0, \ldots, X_{N-1}$. The Karhunen-Loeve transform (KLT) is a unitary transformation U_N, given by the eigenvalue decomposition $K_{X^N} = U_N\Lambda_N U_N^H$, that maps X^N to $Y^N = U_N^H X^N$. This changes its basis to the eigenvectors of its correlation matrix and implies that the correlation matrix of the transformed signal Y^N is diagonal. Let $\lambda_i^{(N)} = [\Lambda_N]_{i,i}$ be the eigenvalues of K_{X^N} for $i = 0, \ldots, N-1$. Since the Frobenius norm $\|A\|_F \triangleq \sqrt{\sum_{i,j}|A_{i,j}|^2}$ is invariant under unitary transformation, it follows that

$$\frac{1}{N}\|K_{X^N}\|_F^2 = \frac{1}{N}\sum_{i=0}^{N-1}\left|\lambda_i^{(N)}\right|^2.$$

Computing the norm directly from the entries of the matrix gives the limit

$$\lim_{N\to\infty} \frac{1}{N} \|K_{X^N}\|_F^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} |r_{m-n}|^2 = \sum_{l=-\infty}^{\infty} |r_l|^2 ,$$

which converges because

$$\sum_{l=-\infty}^{\infty} |r_l|^2 \leq \left(\sum_{l=-\infty}^{\infty} |r_l| \right)^2 < \infty.$$

Since the 2-norm of the diagonal is upper bounded by the Frobenius norm, one can measure how well a transform decorrelates a process by computing how much of the Frobenius norm is contributed by the diagonal.

Lemma 1.4. *The DFT asymptotically decorrelates a WSS process in the sense that the correlation matrix of the output vector $Y^N = F_N X^N$ contains all of its energy on the diagonal as $N \to \infty$. In particular, one finds that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \left| [K_{Y^N}]_{k,k} \right|^2 = \lim_{N\to\infty} \frac{1}{N} \|K_{Y^N}\|_F^2 .$$

Proof. First, we notice that the k-th diagonal element of K_{Y^N} is equal to $S_X\left(\frac{2\pi k}{N}\right)$ and compute

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} [K_{Y^N}]_{k,k}^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \left| S_X\left(\frac{2\pi k}{N}\right) \right|^2 = \frac{1}{2\pi} \int_0^{2\pi} |S_X(\omega)|^2 \, d\omega = \sum_{l=-\infty}^{\infty} |r_l|^2 ,$$

where the second step follows from the continuity of $S_X(\omega)$ and the third from the Parseval relation. Next, we recall that

$$\sum_{l=-\infty}^{\infty} |r_l|^2 = \lim_{N\to\infty} \frac{1}{N} \|K_{X^N}\|_F^2 = \lim_{N\to\infty} \frac{1}{N} \|K_{Y^N}\|_F^2 .$$

□

Theorem 1.5 (Szegö)**.** *Let $S_X(\omega)$ be the power spectral density of a WSS process $\{X_n\}_{n\in\mathbb{Z}}$ and assume that the autocorrelation sequence*

$$c_n^{(\alpha)} \triangleq \frac{1}{2\pi} \int_0^{2\pi} |S_X(\omega)|^{\alpha/2} e^{i\omega n} d\omega$$

is absolutely summable for $\alpha = 1$. Then, for any continuous function $f : [0, M] \to \mathbb{R}$, with $M > \max_{\omega\in[0,2\pi]} \sqrt{S_X(\omega)}$, one finds that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f\left(S_X\left(\frac{2\pi k}{N}\right)\right) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f\left(\lambda_i^{(N)}\right),$$

where $\lambda_i^{(N)}$ denotes the i-th eigenvalue of the autocorrelation matrix K_{X^N}.

Sketch of proof. Since $f(x)$ is continuous, the Weierstrass approximation theorem shows that, for any $\epsilon > 0$, there exists a degree $d < \infty$ polynomial that approximates $f(x)$ on $[0, M]$ in the sense that

$$\sup_{x\in[0,M]} \left| f(x) - \sum_{i=0}^{d} f_i x^i \right| < \epsilon.$$

Therefore, linearity allows us to verify the Theorem's conclusion only for $f(x) = x^i$ with $i \in \mathbb{N}$. Since $c_n^{(\alpha+\beta)} = c_n^{(\alpha)} * c_n^{(\beta)}$, one can use the z-transform to show that $c_n^{(\alpha)}$ is absolutely summable for all $\alpha \in \mathbb{N}$.

For each $\alpha \in \mathbb{N}$, one can define Z_n to be the WSS process with this autocorrelation sequence and let K_{Z^N} be the length-N autocorrelation matrix. Applying the approach taken in Lemma 1.4, one finds that

$$\frac{1}{N}\sum_{i=0}^{N-1}\left[\lambda_i^{(N)}\right]^{\alpha} = \frac{1}{N}\left\|\left(K_{X^N}\right)^{\alpha/2}\right\|_F^2 = \frac{1}{N}\left\|K_{Z^N}\right\|_F^2.$$

Taking the limit exposes the Fourier transform connection

$$\lim_{N\to\infty}\frac{1}{N}\left\|K_{Z^N}\right\|_F^2 = \sum_{l=-\infty}^{\infty}\left|c_l^{(\alpha)}\right|^2 = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\left[S_X\left(\frac{2\pi k}{N}\right)\right]^{\alpha}.$$

□

1.3 Real WSS Processes

One might naturally expect that these results also apply to real WSS processes. While this is true, it is somewhat surprising that the analysis becomes harder rather than easier.

Theorem 1.6. *If $\{X_n\}_{n\in\mathbb{Z}}$ is a real WSS process with mean 0 and the autocorrelation, $r_l = E\left[X_n X_{n+l}\right]$, satisfies $\sum_{l=0}^{\infty}|r_l|\, l < \infty$, then the power spectral density is given by*

$$S_X\left(\frac{2\pi\lfloor\kappa N\rfloor}{N}\right) \triangleq \lim_{N\to\infty} E\left[\left|\sqrt{\frac{2}{N}}\sum_{n=0}^{N-1} X_n \cos\left(\frac{2\pi\lfloor\kappa N\rfloor}{N}n\right)\right|^2\right] = \sum_{l=-\infty}^{\infty} r_n \cos\left(2\pi\kappa l\right)$$

and both limits converge uniformly for all $0<\kappa<\frac{1}{2}$. Moreover, the result is unchanged if the cos *in the middle term is replaced by* sin.

Proof. We start by defining

$$\hat{X}_N\left(\frac{2\pi k}{N}\right) \triangleq \sqrt{\frac{2}{N}}\sum_{n=0}^{N-1} X_n\cos\left(\frac{2\pi k}{N}n\right)$$

and observing that

$$\begin{aligned}
E\left[\left|\hat{X}_N\left(\frac{2\pi k}{N}\right)\right|^2\right] &= E\left[\left(\sqrt{\frac{2}{N}}\sum_{n=0}^{N-1}X_n\cos\left(\frac{2\pi k}{N}n\right)\right)\left(\sqrt{\frac{2}{N}}\sum_{m=0}^{N-1}X_m\cos\left(\frac{2\pi k}{N}m\right)\right)\right] \\
&= \frac{2}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\cos\left(\frac{2\pi k}{N}n\right)\cos\left(\frac{2\pi k}{N}m\right)E\left[X_n X_m\right] \\
&= \frac{2}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\frac{1}{2}\left(\cos\left(\frac{2\pi k}{N}(m-n)\right)+\cos\left(\frac{2\pi k}{N}(m+n)\right)\right)r_{m-n} \\
&= \frac{1}{N}\sum_{l=-N-1}^{N-1}(N-l)\cos\left(\frac{2\pi k}{N}l\right)r_l + \frac{1}{N}\sum_{l=-N-1}^{N-1}\sum_{n=\min\{0,-l\}}^{\max\{N-1,N-l-1\}}\cos\left(\frac{2\pi k}{N}(2n+l)\right)r_l \\
&= \frac{1}{N}\sum_{l=-N-1}^{N-1}(N-l)\cos\left(\frac{2\pi k}{N}l\right)r_l + \frac{1}{N}\sum_{l=1}^{N-1}\sum_{n=0}^{N-l-1}\cos\left(\frac{2\pi k}{N}(2n+l)\right)r_l + \frac{1}{2N}\sum_{l=1}^{N-1}\sum_{n=l}^{N-1}\cos\left(\frac{2\pi k}{N}(2n-l)\right)r_l \\
&= \frac{1}{N}\sum_{l=-N-1}^{N-1}(N-l)\cos\left(\frac{2\pi k}{N}l\right)r_l + \frac{2}{N}\sum_{l=1}^{N-1}r_l\sum_{n=0}^{N-l-1}\cos\left(\frac{2\pi k}{N}(2n+l)\right) \\
&= \sum_{k=-N-1}^{N-1}\left(1-\frac{l}{N}\right)\cos\left(\frac{2\pi k}{N}l\right)r_l - \frac{2}{N}\sum_{l=1}^{N-1}r_l l\left(1+O\left(\frac{l^2}{N^2}\right)\right) \quad \text{if } 0<k<N/2 \\
&= \sum_{k=-N-1}^{N-1}\left(1-\frac{l}{N}\right)\cos\left(\frac{2\pi k}{N}l\right)r_l - \frac{2}{N}\sum_{l=1}^{N-1}r_l l\left(1+O\left(\frac{l^2}{N^2}\right)\right),
\end{aligned}$$

where the last term is negligible because

$$\frac{2}{N}\left|\sum_{l=1}^{N-1} r_l l\left(1+O\left(\frac{l^2}{N^2}\right)\right)\right| \le \frac{2}{N}\sum_{l=1}^{N-1}|r_l|\, l\left(1+O\left(\frac{l^2}{N^2}\right)\right) \le O(1)\frac{2}{N}\sum_{l=1}^{\infty}|r_l|\, l = O\left(\frac{1}{N}\right)$$

because $\sum_{l=1}^{\infty}|r_l|\, l < \infty$. For $0 < k < N/2$, the estimate $\sum_{n=0}^{N-l-1}\cos\left(\frac{2\pi k}{N}(2n+l)\right) = -l\left(1+O\left(\frac{l^2}{N^2}\right)\right)$ follows from computing the sum and then performing a series expansion in N^{-1}. Finally, we can choose $k_N = \lfloor \kappa N \rfloor$ and see that convergence follows from the continuity of $S_X(\omega)$, which holds because r_l is absolutely summable and $\left|\left(1-\frac{l}{N}\right)\cos\left(\frac{2\pi k}{N}l\right) r_l\right| \le |r_l|$ whenever it appears in the sum. Also, the argument above is essentially identical for $\sqrt{\frac{2}{N}}\sum_{n=0}^{N-1} X_n \sin\left(\frac{2\pi k}{N}n\right)$ because

$$2\sin(u)\sin(v) = \cos(u-v) - \cos(u+v).$$

□

Corollary 1.7. *The discrete Fourier sin/cos basis vectors asymptotically decorrelate a real WSS random process. For even N, the vectors are given by the columns of the $N \times N$ orthogonal matrix*

$$[A_N]_{n,k} = \begin{cases} \sqrt{\frac{1}{N}} & \text{if } k = 0 \\ \sqrt{\frac{2}{N}}\sin\left(\frac{\pi(k+1)}{N}n\right) & \text{if } k = 1,3,\ldots,N-3 \\ \sqrt{\frac{2}{N}}\cos\left(\frac{\pi k}{N}n\right) & \text{if } k = 2,4,\ldots,N-2 \\ \sqrt{\frac{1}{N}}\cos(\pi n) & \text{if } k = N-1 \end{cases}$$

Proof. The proof uses arguments similar to Lemma 1.4. □

Example 1.8. For a discrete-time communication channel with real WSS Gaussian noise, one can achieve capacity by signaling on the basis vectors described in Corollary 1.7. Using waterfilling with level ν, the achievable rate on each subchannel is given by

$$R_k = \frac{1}{2}\log\left(1+\frac{\left(\nu - S_X\left(\frac{\pi k}{N}\right)\right)^+}{S_X\left(\frac{\pi k}{N}\right)}\right) + o_N(1),$$

for $k = 0,\ldots,N-1$. Therefore, the average rate as $N \to \infty$ is given by

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} R_k = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{2}\log\left(1+\frac{\left(\nu - S_X\left(\frac{\pi k}{N}\right)\right)^+}{S_X\left(\frac{\pi k}{N}\right)}\right) + o_N(1) = \frac{1}{\pi}\int_0^{\pi}\frac{1}{2}\log\left(1+\frac{\left(\nu - S_X(\omega)\right)^+}{S_X(\omega)}\right)d\omega,$$

for an average input power of

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} P_k = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\left(\nu - S_X\left(\frac{\pi k}{N}\right)\right)^+ = \frac{1}{\pi}\int_0^{\pi}\left(\nu - S_X(\omega)\right)^+ d\omega.$$

In fact, the natural generalization of Theorem 1.5 can be used to prove equals the capacity.