The Peeling Decoder : Theory and some Applications

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Thanks to Henry Pfister, Avinash Vem, Nagaraj Thenkarai Janakiraman, Kannan Ramchandran, Jean-Francois Chamberland

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Introduction

Message passing algorithms

- Remarkably successful in coding theory
- Used to design capacity-achieving codes/decoders for a variety of channels
- Tools have been developed to analyze their performance

Two main goals

Goal 1

Review some developments in modern coding theory and show how to analyze the performance of a simple peeling decoder for the BEC and p-ary symmetric channels.

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Review some developments in modern coding theory and show how to analyze the performance of a simple peeling decoder for the BEC and p-ary symmetric channels.

Goal 2

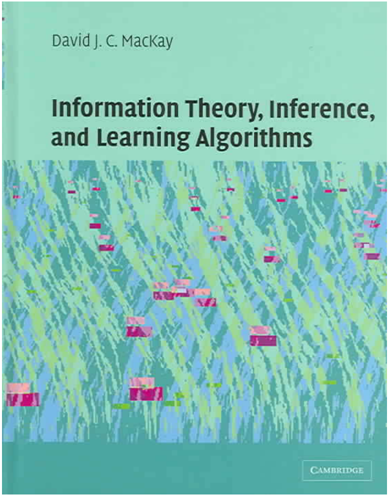
Show that the following problems have the same structure as channel coding problems and show how to use the peeling decoder to solve them.

Problems

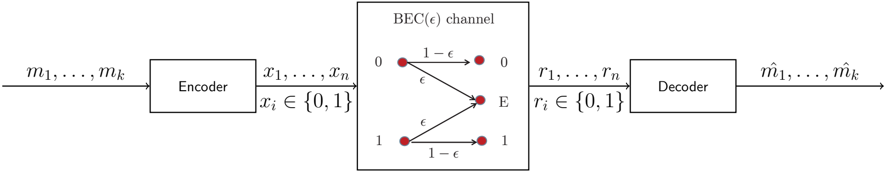
- Uncoordinated massive multiple access
- Sparse Fourier transform (SFT) computation
- Sparse Walsh-Hadamard transform computation
- Compressed sensing
  - Data stream computing
  - Group testing
  - Compressive phase retrieval

Remembering Sir David MacKay

David Mackay's rediscovery of LDPC codes and his very interesting book on Information Theory has undoubtedly had a big influence on the field.



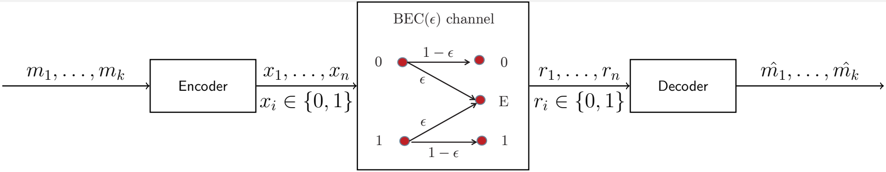
Binary erasure channel (BEC) and erasure correction



Channel coding problem

- Transmit a message $\underline{m} = [m_1, \ldots, m_k]^T$ through a binary erasure channel
- Encode the k-bit message $\underline{m}$ into a n-bit codeword $\underline{x}$
- Redundancy is measured in terms of rate of the code $R = k/n$

Binary erasure channel (BEC) and erasure correction



Capacity achieving sequence of codes

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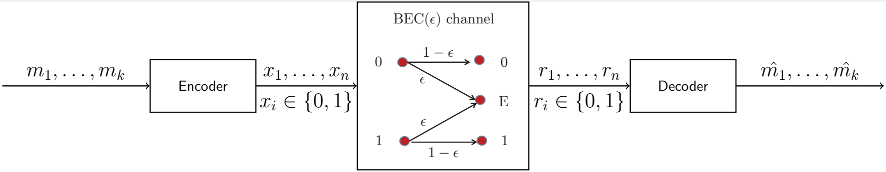

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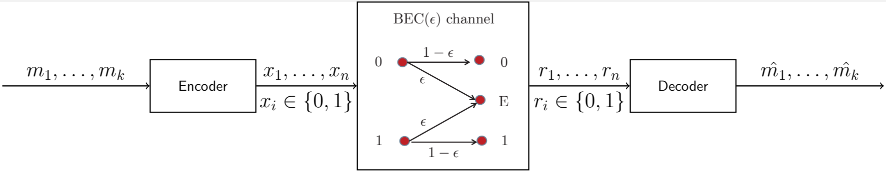

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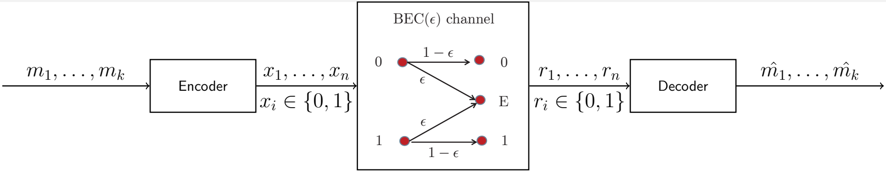
- Probability of erasure P_e^n
- Rate R^n
- Capacity achieving if $P_e^n \to 0$ as $n \to \infty$ while $R^n \to C$

Binary erasure channel (BEC) and erasure correction



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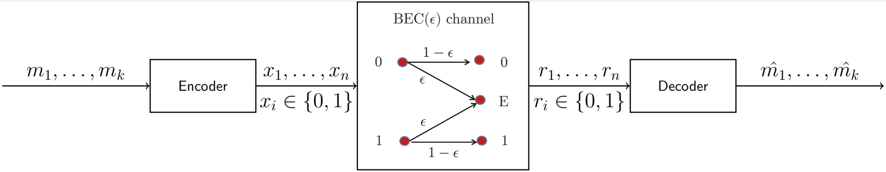

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Significance of the erasure channel

- Introduced by Elias in 1954 as a toy example
- Has become the canonical model for coding theorists to gain insight

(n, k) Binary linear block codes - basics

$\mathbf{G}$ is a $n \times k$ generator matrix

$$\begin{bmatrix} g_{1,1} & \cdots & g_{k,l} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ g_{n,1} & & g_{k,l} \end{bmatrix} \begin{bmatrix} m_1 \\ \vdots \\ m_k \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

Example - (6,3) code

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

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Parity check matrix - $\mathbf{H}$ is a $(n-k) \times n$ matrix s.t. $\mathbf{HG} = \mathbf{0} \Rightarrow \mathbf{H}\underline{x} = 0$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Tanner graph representation of codes

$$\begin{array}{c} \begin{array}{cccccc} x_1, & x_2, & x_3, & x_4, & x_5, & x_6 \end{array} \\ H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$x_1 \oplus x_3 \oplus x_4 = 0$$

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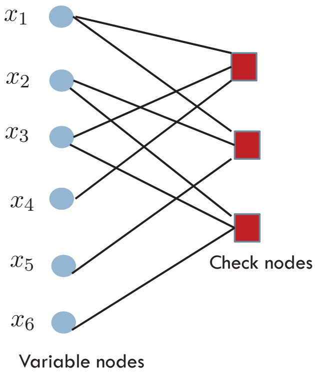

- Gallager'63, Tanner'81
- Parity check matrix implies that $\mathbf{H}\underline{x} = 0$
- Code constraints can be specified in terms of a bipartite (Tanner) graph

Peeling decoder for the BEC

$$H = \begin{array}{c} x_1, x_2, x_3, x_4, x_5, x_6 \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

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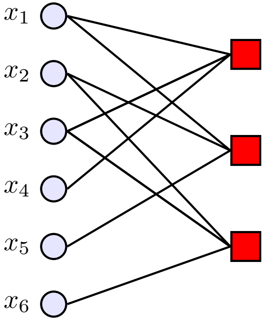

Tanner Graph

- Zyablov and Pinsker'74, Luby et al '95
- Remove edges incident on known variable nodes and adjust check node values
- If there is a check node with a single edge, it can be recovered

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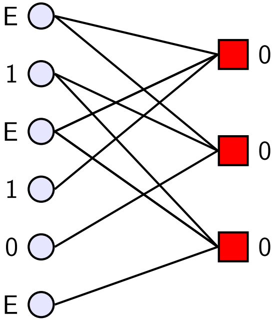

Received block

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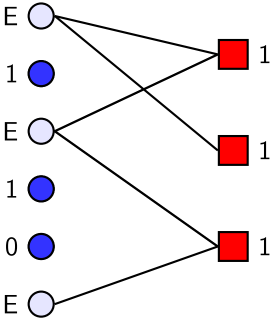

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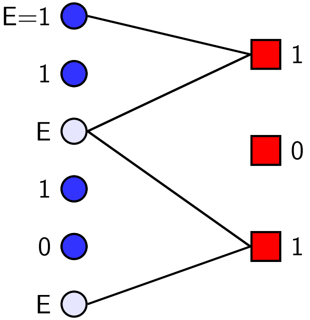

Peeling Step 2

- Zyablov and Pinsker'74, Luby et al '95
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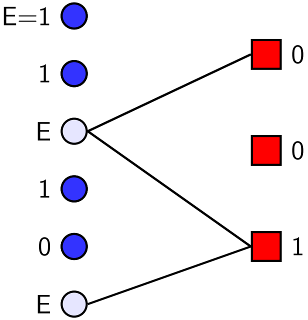

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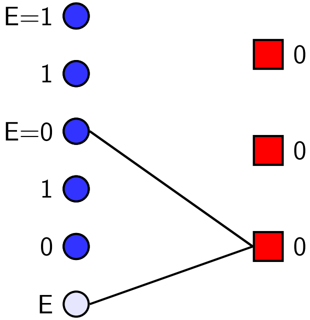

Peeling Step 3

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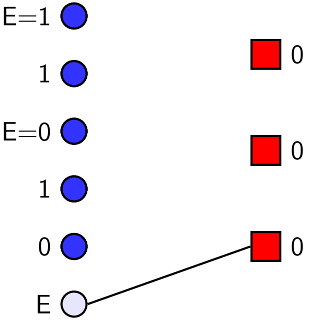

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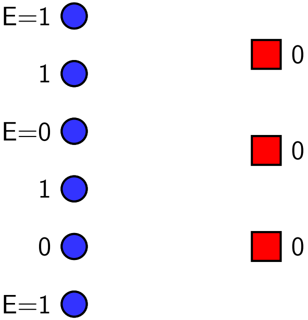

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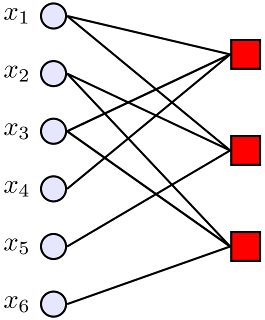
Message passing decoder for the BEC



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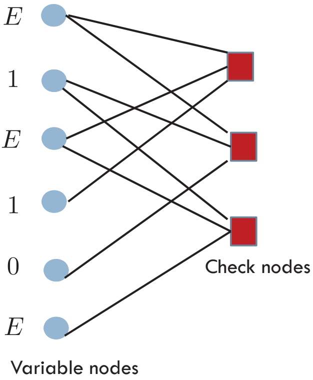

- Pass messages between variable nodes and check nodes along the edges
- Messages $\in$ {value of var node (NE), erasure (E)}
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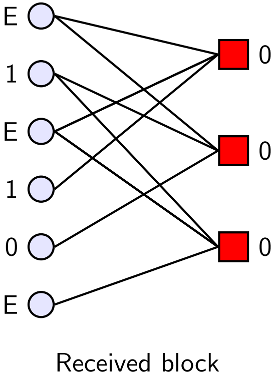

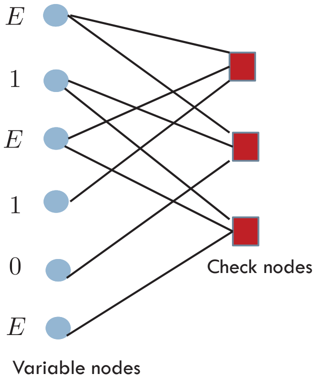

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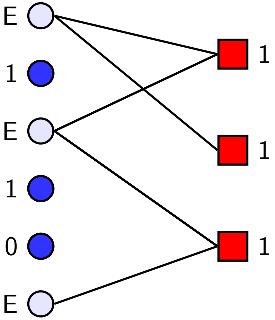

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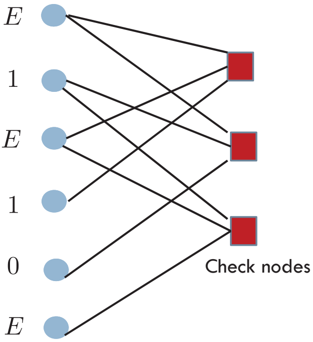

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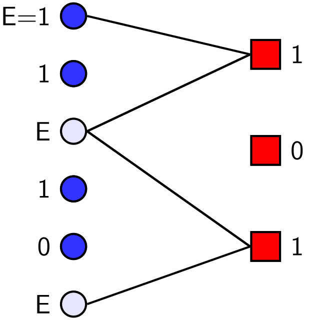
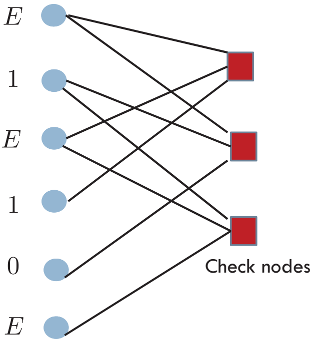


Peeling Step 2



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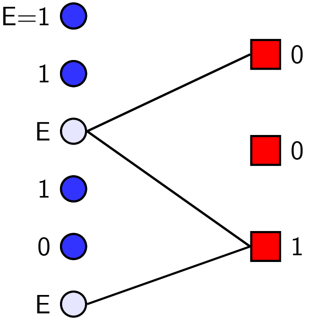

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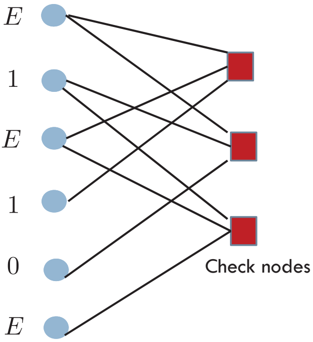

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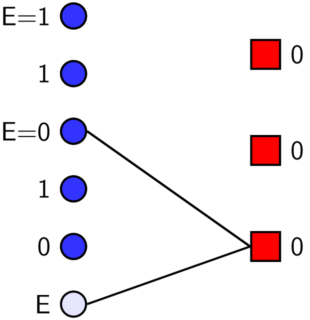

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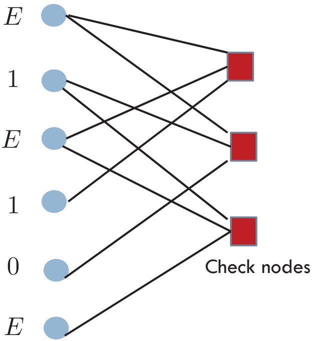

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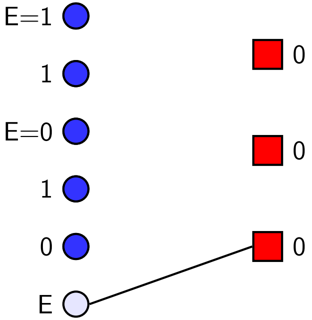

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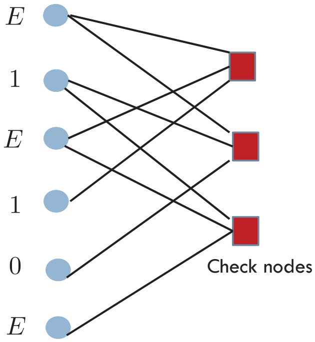

Variable nodes

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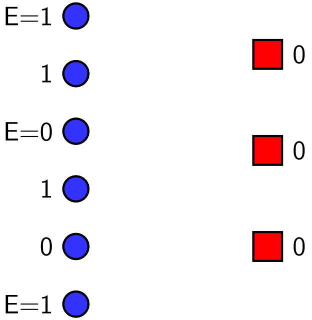
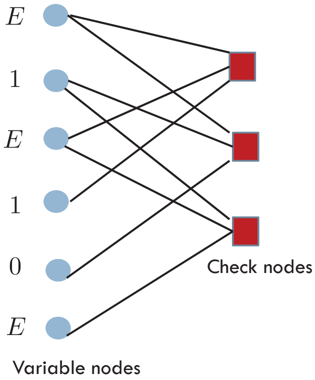


Peeling Step 4



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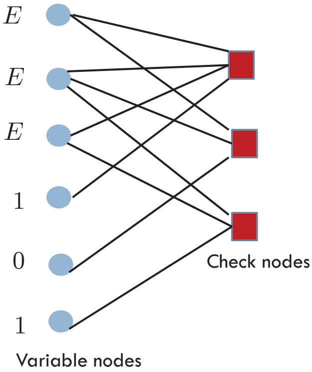
Peeling decoder is a greedy decoder

$$\mathbf{H} = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\begin{aligned} x_1 \oplus x_1 \oplus x_3 \oplus x_4 &= 0 \\ x_1 \oplus x_2 \oplus x_5 &= 0 \\ x_2 \oplus x_3 \oplus x_6 &= 0 \end{aligned}$$



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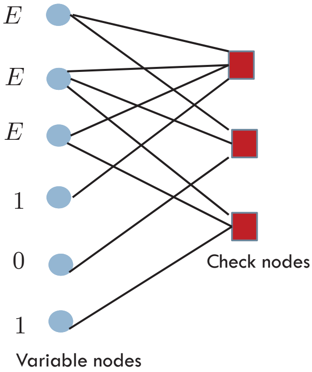
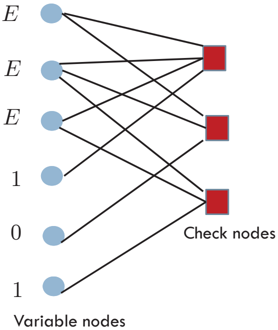


Linearly independent set of equations

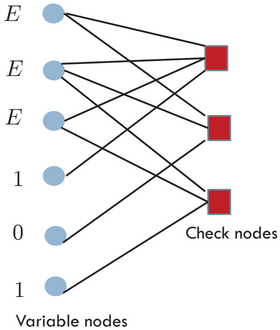
$$\begin{aligned} x_1 \oplus x_1 \oplus x_3 &= x_4 \\ x_1 \oplus x_2 &= x_5 \\ x_2 \oplus x_3 &= x_6 \end{aligned}$$

Degree distributions



- VN d.d. from node perspective - $L(x) = \sum_i L_i x^i = \frac{3}{6}x + \frac{2}{6}x^2 + \frac{1}{6}x^3$

Degree distributions



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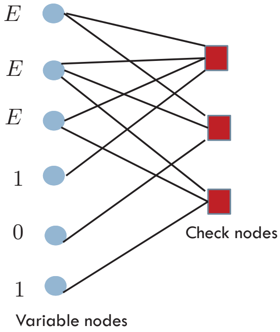
Degree distributions



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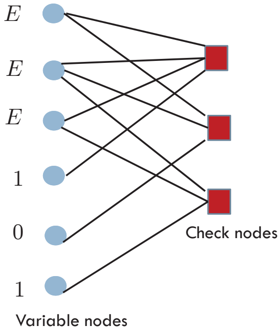
Degree distributions



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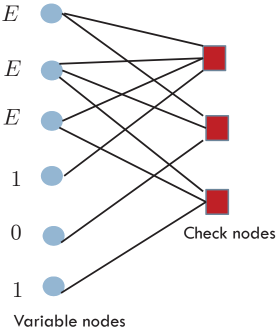
Degree distributions



- Rate - $r(\lambda, \rho) = 1 - \frac{l_{\text{avg}}}{r_{\text{avg}}} = 1 - \frac{\int_0^1 \rho(x)\ dx}{\int_0^1 \lambda(x)\ dx}$

- VN d.d. from node perspective - $L(x) = \sum_i L_i x^i = \frac{3}{6}x + \frac{2}{6}x^2 + \frac{1}{6}x^3$
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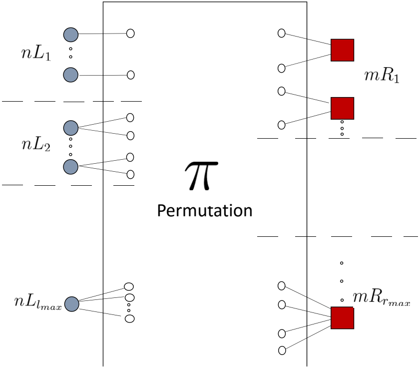
LDPC code ensemble



LDPC(n, λ, ρ) ensemble

- Ensemble of codes obtained by using different permutations π
- Assume there is only one edge between every var node and check node
- For every n, we get an ensemble of codes with the same (λ, ρ)
- Low density parity check (LDPC) ensemble if graph is of low density

Analysis of the message passing decoder

- If we pick a code uniformly at random from the LDPC(n, λ, ρ) ensemble and use it over a BEC(ϵ) with l iterations of message passing decoding, what will be the probability of erasure P_e^n in the limit $l, n \to \infty$?

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Relevant literature

- Papers by Luby, Mitzenmacher, Shokrollahi, Spielman, Stemann 97-'02
- Explained in Modern coding theory by Richardson and Urbanke
- Henry Pfister's course notes on his webpage

Analysis of the message passing decoder

Computation graph

Computation graph $\mathcal{C}_l(x_1, \lambda, \rho)$ of bit x_1 of depth l (l-iterations) is the neighborhood graph of node x_1 of radius l.

Analysis of the message passing decoder

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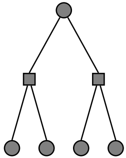
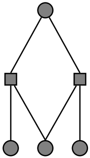


$1 - O(1/n)$



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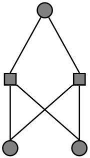

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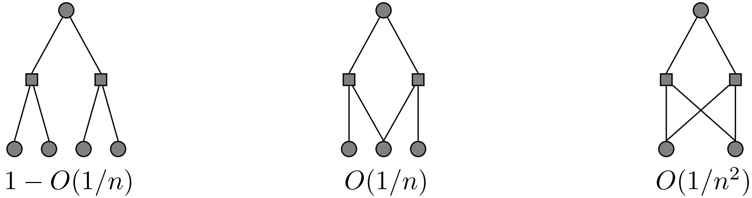


Computation tree

For fixed (l_{max}, r_{max}), in the limit of large block lengths a computation graph of depth-l looks like a tree with high probability

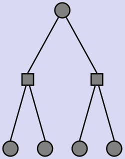
Analysis of the message passing decoder

Computation Tree Ensemble-$\mathcal{T}_l(\lambda, \rho)$

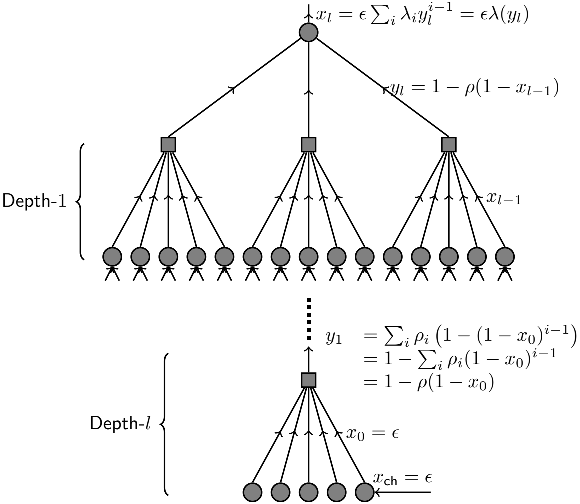
Ensemble of bipartite trees of depth l rooted in a variable node (VN) where

- Root node has i children(CN's) with probability L_i
- Each VN has i children(CN's) with probability λ_i
- Each CN has i children(VN's) with probability ρ_i

Example: $\mathcal{C}_{l=1}(\lambda(x) = x, \rho(x) = x^2)$



Density evolution



Recall

- $\rho(x) = \sum_i \rho_i x^{i-1}$
- $\sum_i \rho_i = 1$
- $\lambda(x) = \sum_i \lambda_i x^{i-1}$
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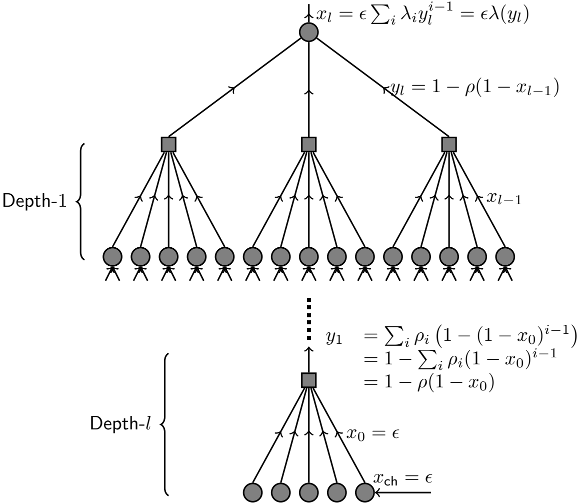
Recursion

$$x_0 \quad = \quad \epsilon$$

Density evolution



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Recursion

$$
\begin{aligned}
x_0 &= \epsilon \\
y_l &= 1 - \rho(1 - x_{l-1})
\end{aligned}
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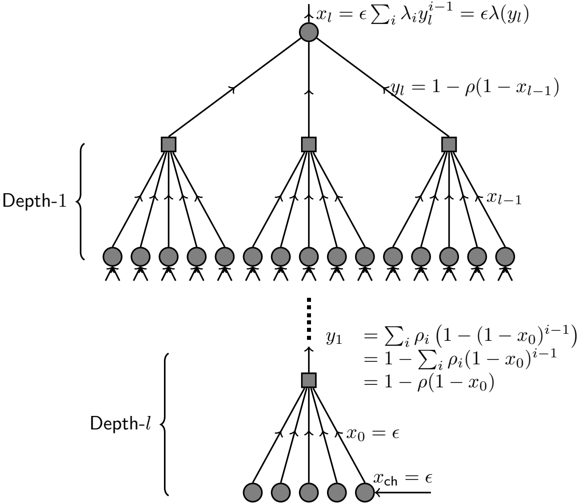
Density evolution



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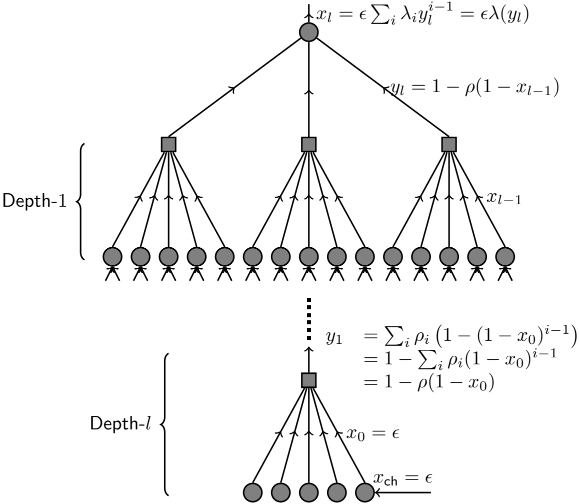
Density evolution



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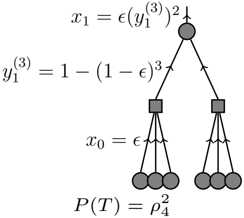
$\lambda(x) = x^2, \rho(x) = \rho_4 x^3 + \rho_5 x^4$



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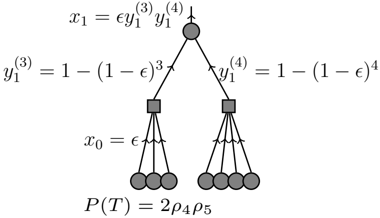

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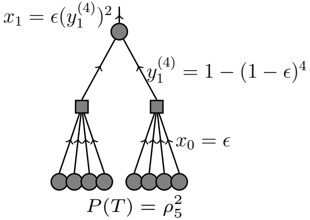

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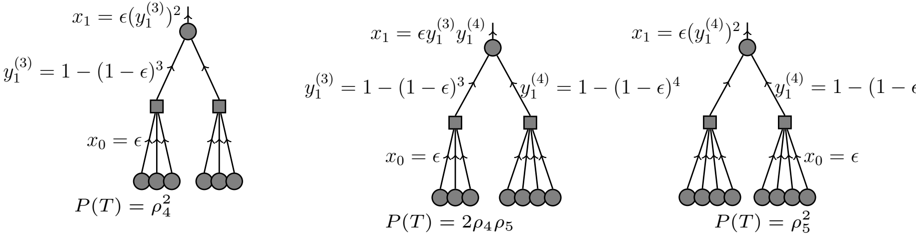

$$
\begin{aligned}
\mathbb{E}_{\mathsf{LDPC}(\lambda,\rho)}[x_1] &= \sum_{T\in\mathcal{T}_1(\lambda,\rho)} P(T) * x_1(T,\epsilon) \\
&= \epsilon(\rho_4 y_1^{(3)} + \rho_5 y_1^{(4)})^2 \\
&= \epsilon(1-\rho_4(1-\epsilon)^3 - \rho_5(1-\epsilon)^4)^2 \\
&= \epsilon\lambda\left(1-\rho(1-\epsilon)\right)
\end{aligned}
$$

Threshold

Convergence condition

$$x_l = \epsilon\lambda(1-\rho(1-x_{l-1})) = f(\epsilon, x_{l-1})$$

$$\begin{aligned} x_l \text{ converges to 0 if} \quad & f(\epsilon,x) < x,\ \ x \in (0,\epsilon] \\ \text{There is a fixed point if} \quad & f(\epsilon,x) = x,\ \ \text{for some } x \in (0,\epsilon] \end{aligned}$$

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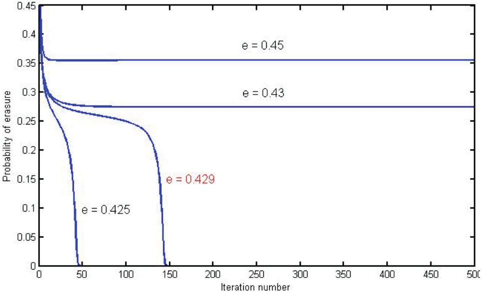

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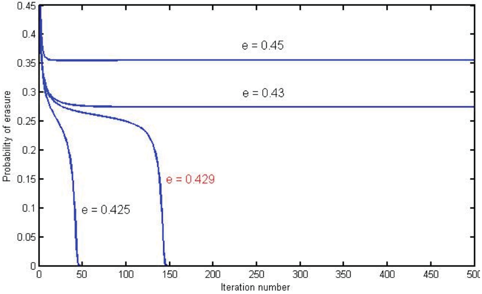


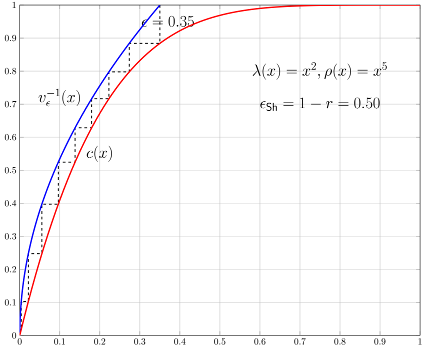
The threshold $\epsilon^{\mathsf{BP}}(\lambda, \rho)$ is defined as

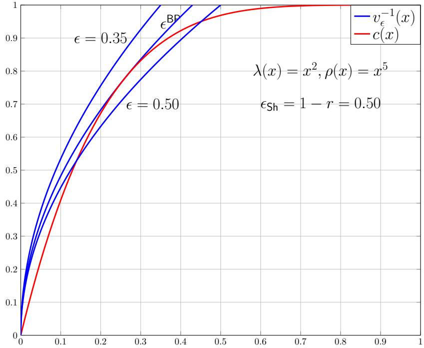
$$\epsilon^{\mathsf{BP}}(\lambda, \rho) = \sup\{\epsilon \in [0, 1] : x_l \to 0 \text{ as } l \to \infty\}$$

Exit charts - Ashikmin, Kramer, ten Brink'04

Node functions

- Var node function: $v_\epsilon(x) = \epsilon\lambda(x)$
- Check node function: $c(x) = 1 - \rho(1 - x)$





Optimality of EXIT chart matching

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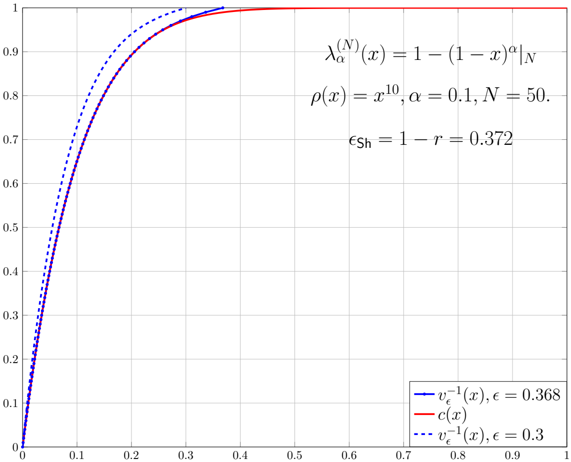

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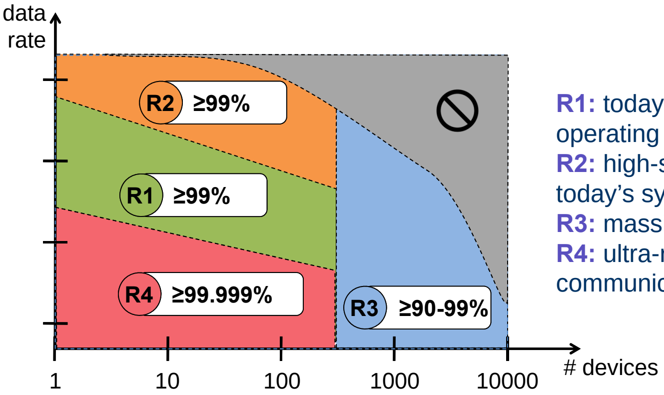
Summary

- Understand what degree distributions $(\lambda(x), \rho(x))$ mean
- Given a (λ, ρ) and ϵ, what will be the P_e^n as $l, n \to \infty$?
- Can you compute the threshold?
- Is a $(\lambda(x), \rho(x))$ pair optimal?

Application 1

The changing mobile landscape

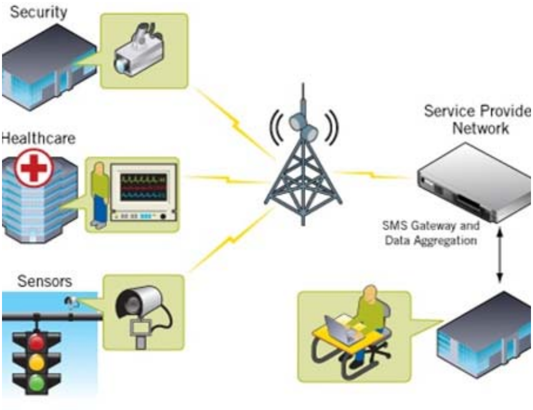
- 5G will not only be “4G but faster” but will support new models such as IoT
- Current wireless - a few devices with sustained connectivity
- Future wireless - massive no. of devices requesting sporadic connectivity



R1: today’s systems operating region
R2: high-speed versions of today’s systems
R3: massive access
R4: ultra-reliable communication

The changing mobile landscape

- Current wireless - a few devices with sustained connectivity
- Future wireless - many uncoordinated devices requesting sporadic connectivity



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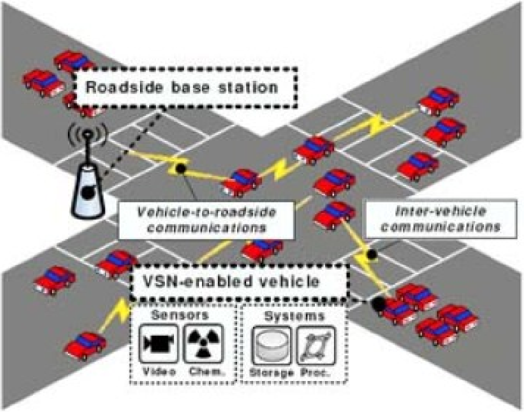


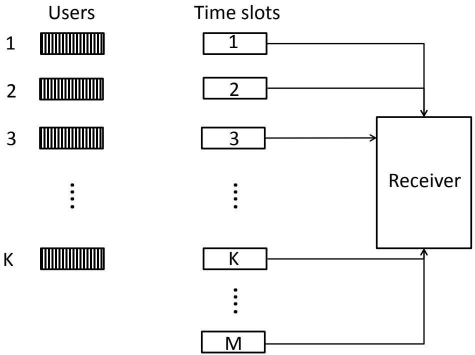
A possible MAC frame structure

- Total of Q users out of which K are active
- Q is very large and K is a small fraction of Q

- Beacon is used to obtain coarse synchronization
- Each user transmits a signature sequence
- BS estimates the no. of users (K) (Chen, Guo '14, Calderbank)
- Picks an M and broadcasts it

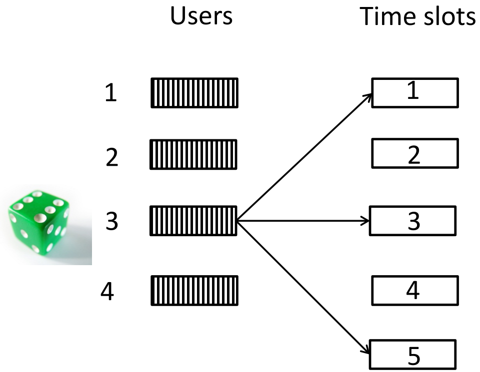
System under consideration

- Wireless network with K distributed users (no coordination)
- Each user has one packet of info to transmit to a central receiver
- Total time is split into M slots (packet duration)
  - Some policy used to decide if they transmit in j-th slot or not
  - Receiver knows the set of users transmitting in the j-th slot



Random access paradigm

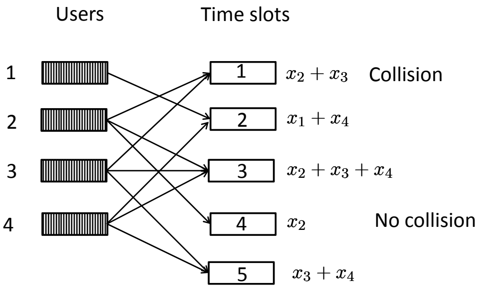
- k-th user:
  - Generates a random variable $D_k \in \{1, \dots, M\}$
  - Generating PMF is f_D, i.e., $Pr(D_k = i) = f_D[i]$
  - Transmits during D_k time slots drawn uniformly from $\{1, \dots, M\}$



- In this example, $D_3 = 3$ and user 3 transmits in slots $\{1, 3, 5\}$

Iterative interference cancelation

- If exactly one user transmits per slot, then packet is decoded w.h.p.
- If more than one user transmits per slot, then collision
  - Rx subtracts previously decoded packets from collided packets
  - If Rx can subtract all but one, remaining packet is decoded w.h.p.
  - Otherwise, the received packet is saved for future processing
  - Once all K packets recovered, an ACK terminates the transmission
  - Similar to interference cancellation in multi-user detection



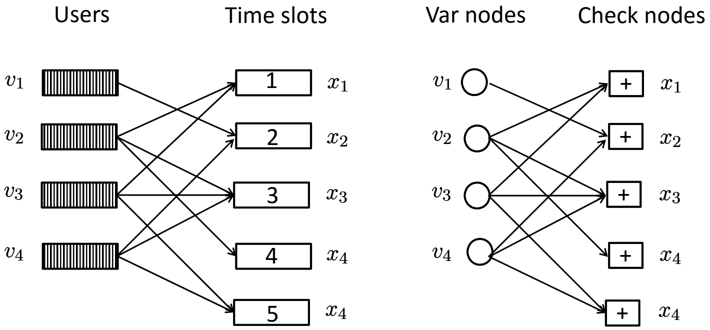
Performance measure - Efficiency

- Suppose M time slots needed to successfully transmit all K packets
- Then, the efficiency of the system is said to be

$$\eta = K/M \text{ packets/slot}$$

Graphical representation (Liva 2012)

- Tanner graph representation for the transmission scheme
- Variable nodes $\leftrightarrow$ users, Check nodes $\leftrightarrow$ received packets
- Message-passing decoder - peeling decoder for the erasure channel



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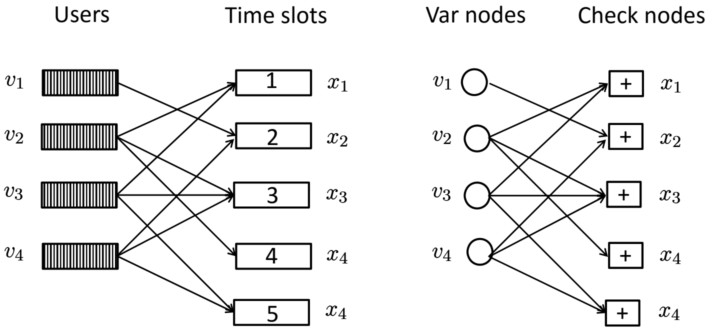
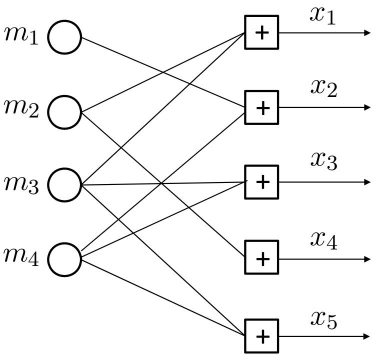


- L_i (R_i) - fraction of left (right) nodes with degree i - notice that $\boxed{L_i = f_D[i]}$
- λ_i (ρ_i) - fraction of edges connected to left (right) nodes with deg i

Low density generator matrix (LDGM) codes



- $L(x) = \frac{1}{4}x + \frac{1}{4}x^2 + \frac{1}{2}x^3$
- $\lambda(x) = \frac{1}{9} + \frac{2}{9}x + \frac{6}{9}x^2$
- $R(x) = \frac{1}{5}x + \frac{4}{5}x^2$
- $\rho(x) = \frac{1}{9} + \frac{8}{9}x$
- Rate $R = \frac{\int_0^1 \lambda(x)\ dx}{\int_0^1 \rho(x)\ dx}$

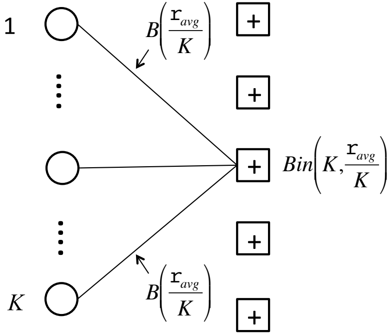
DE for LDPC

$$\begin{aligned} x_0 &= \epsilon \\ y_l &= 1 - \rho(1 - x_{l-1}) \\ x_l &= \epsilon\lambda(y_l) \\ x_l &= \epsilon\lambda(1 - \rho(1 - x_{l-1})) \end{aligned}$$

DE for LDGM

$$\begin{aligned} x_0 &= 1 \\ y_l &= 1 - \rho(1 - x_{l-1}) \\ x_l &= \lambda(y_l) \\ x_l &= \lambda(1 - \rho(1 - x_{l-1})) \end{aligned}$$

Poisson approximation for check node d.d.



Slot transmission probability

User k transmits in slot m with prob. $p = \sum_{i=1}^{\infty} L_i \frac{i}{M} = \frac{\mathtt{l}_{avg}}{M} = \frac{\mathtt{r}_{avg}}{K}$

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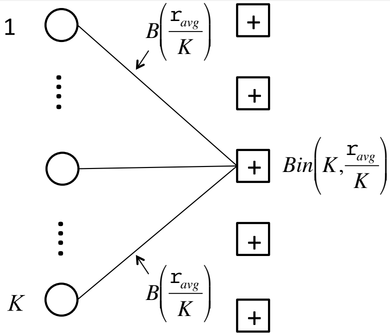

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Optimal multiple access policy

- Poisson approximation for $R(x)$ as $K, M \to \infty$
- Finding optimal f_D - same as finding optimal $\lambda(x)$ for $\rho(x) = e^{-\mathtt{r}_{avg}(1-x)}$

Intuition behind the main result (Narayanan,Pfister'12)

Convergence condition : $\rho(1-\lambda(y)) > 1-y$

$$\rho(1-\lambda(y)) = 1-y$$

$$e^{-\mathtt{r}_{avg}\lambda(y)} = e^{\ln(1-y)}$$

$$\Rightarrow -\mathtt{r}_{avg}\lambda(y) = \ln(1-y) = -\sum_{i=1}^{\infty}\frac{y^i}{i}$$

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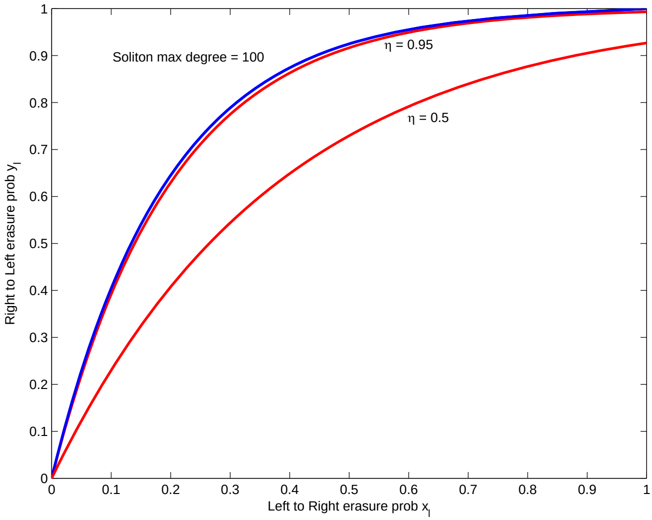
$$\mathtt{r}_{avg}\lambda_i = \frac{1}{i}$$
$$\sum_i \lambda_i = 1 \Rightarrow \boxed{\mathtt{r}_{avg} = \sum_i \frac{1}{i}, \lambda_i = \frac{1/i}{\sum_i 1/i}} \Rightarrow \boxed{L_i = \frac{1}{i(i-1)}, i \geq 2}$$

Graphical interpretation - EXIT chart



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Optimal distribution is soliton: $f_D[i] = \frac{1}{i(i-1)}$

No. of times	1	2	3	4	...	M
Fraction of users	$\frac{1}{M}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$	...	$\frac{1}{M(M-1)}$

Balls in bins

- M balls thrown into N bins uniformly at random
- If every bin has to be non-empty with prob $1 - \delta$, how large should M be ?

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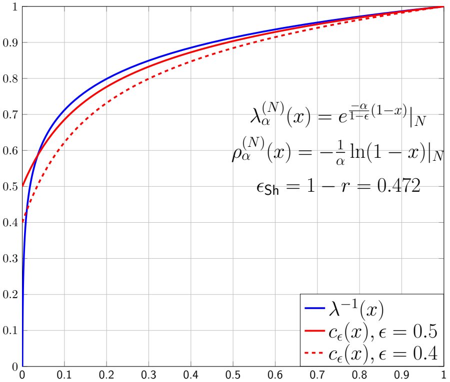
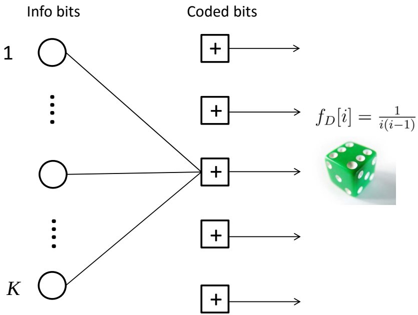
- If every bin has to be non-empty with prob $1 - \delta$, how large should M be ?
  $\boxed{N \log \frac{N}{\delta}}$
- For the multiple access problem, an empty bin means a wasted time slot
- Note that for the soliton the average number of edges is indeed $N \log N$)

Poisson, soliton pair is optimal for rateless codes

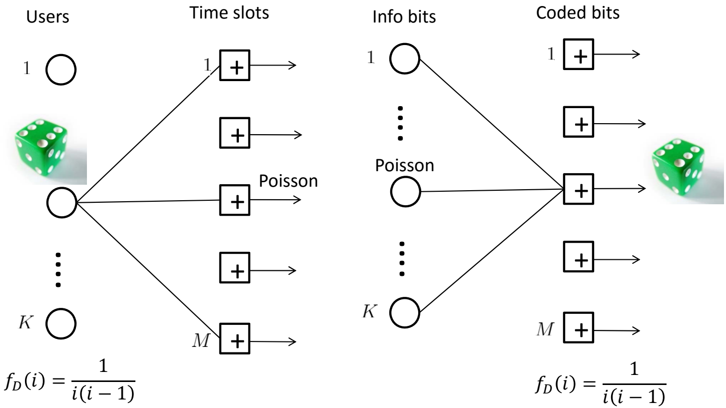




- $x = \lambda(1 - (1-\epsilon)\rho(1-x))$
- $\lambda(x) = e^{-\frac{\alpha}{1-\epsilon}(1-x)}$, optimal right degree is soliton: $\rho(x) = -\frac{1}{\alpha}\ln(1-x)$

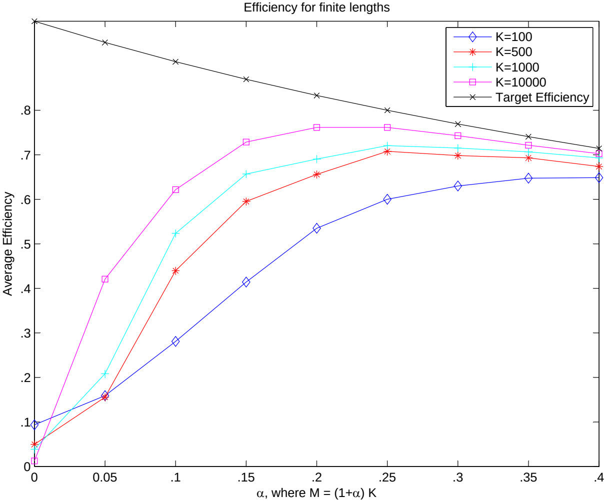
Degree of nodes	1	2	3	4	...	i	...	K
Fraction: $f_D[i]$	$\frac{1}{K}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$	...	$\frac{1}{i(i-1)}$	...	$\frac{1}{K(K-1)}$

Connection with Luby Transform (LT) codes



- For rateless codes $\lambda(x)$ is Poisson and $\rho(x)$ is soliton
- For multiple access $\rho(x)$ is Poisson, optimal $\lambda(x)$ is soliton
- Our result shows that both are optimal pairs

Simulation Results



- Even for $K = 10000$, efficiency close to 0.8 can be obtained

Some open problems

- Fundamental limits on universal multiple access, i.e. K, ϵ not known
- Uncoordinated multiple access with power constraint and Gaussian noise
  - Power penalty for repeating information $\log n$ times on the average
  - Can we achieve the equal rate point on the MAC region with simple decoding?

Back to theory: from erasures to errors

Finite field with p elements

p is prime

- $\mathbb{F}_p - \{0, 1, 2, \ldots, p-1\}$
- $a \oplus b = (a+b) \mod p$
- $a \odot b = (ab) \mod p$
- We can $+, \times, \div$, inverses
- W is a (primitive) element such that $1, W, W^2, \ldots, W^{p-1}$ are distinct

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Example $\mathbb{F}_5$

- $W = 2$
- $W^0 = 1, W^1 = 2, W^2 = 4, W^3 = 3$

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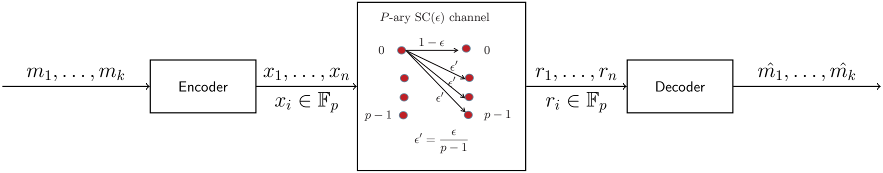
p need not be prime

- Everything can be extended to finite fields with $q = 2^r$ elements
- May be extended to integers - not sure

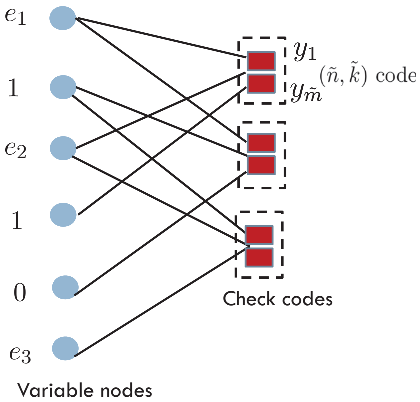
p-symmetric channel and error correction



Error correction coding

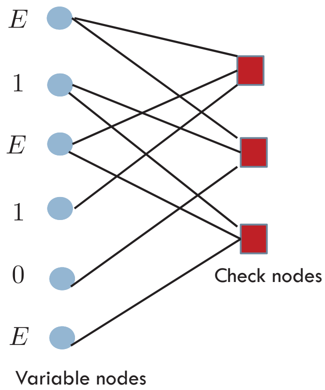
- Another simple channel model which has been extensively considered
- Has been the canonical model for algebraic coding theorists

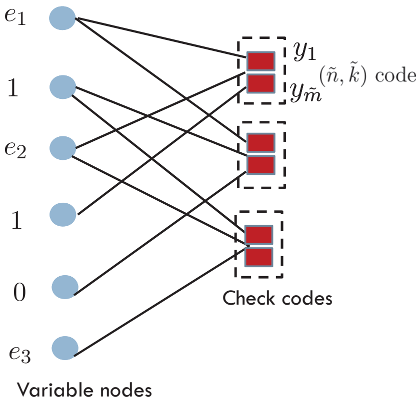
Generalized LDPC code and error channels



- GLDPC introduced by Tanner in 1981
- Each check is a $(\tilde{n}, \tilde{k})$, t-error correcting code
- If there are $\leq t$ errors in a check, it can be recovered
- For now, assume no miscorrections

Peeling process is same for erasure and error channels





- Assume 1-error correcting check code and no miscorrections
- One-to-one correspondence between messages passed - DE can be used
- Not optimal for the error channel but it is not bad at high rates
- Spatially coupled versions are optimal at high rates (Jian, Pfister and N)

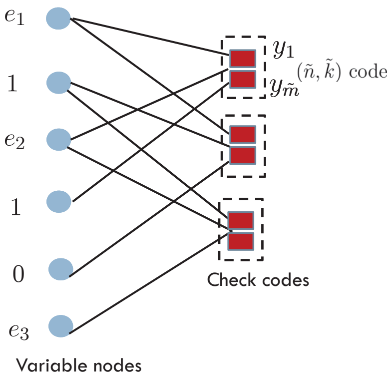
Erasures to errors - tensoring and peeling

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\otimes$$

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 & W^5 \end{bmatrix}$$

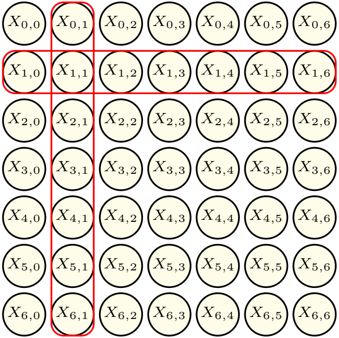
$$\tilde{\mathbf{H}} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & W^2 & W^3 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & W & 0 & 0 & W^4 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & W & W^2 & 0 & 0 & W^5 \end{bmatrix}$$



- W is a primitive element in the field
- Each check is a 1-error correcting code
- If there is exactly one error in a check, it can be recovered

Product code

- Special case of generalized LDPC code
- Let component code $\mathcal{C}$ be an $(\tilde{n}, \tilde{k}, \tilde{d}_{\text{min}})$ linear code
- Well-known that $\mathcal{P}$ is an $(\tilde{n}^2, \tilde{k}^2, \tilde{d}_{\text{min}}^2)$ linear code



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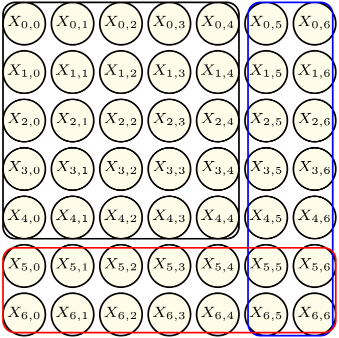
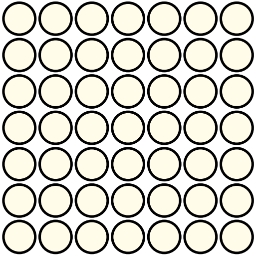
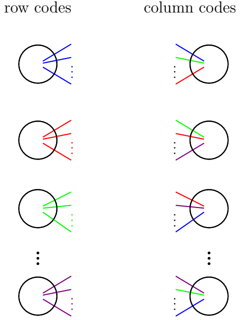


Peeling decoding of product codes

- Hard-decision "cascade decoding" by Abramson in 1968
- Identical to a peeling decoder
- Example: $t = 2$-error-correcting codes, bounded distance decoding





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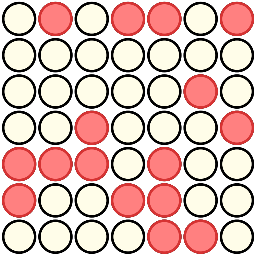
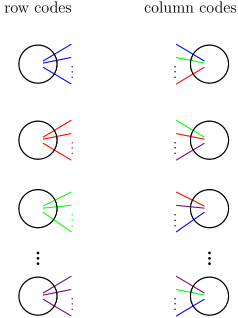


Received block



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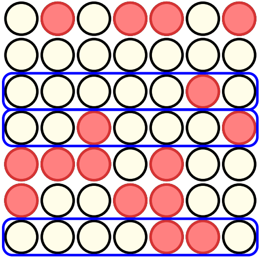
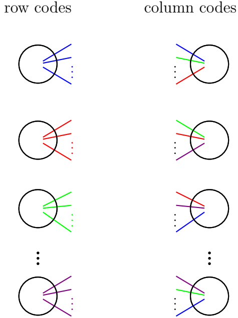

Row decoding



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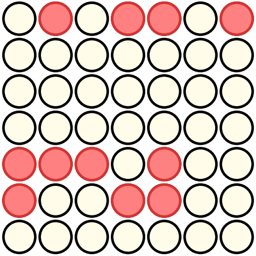

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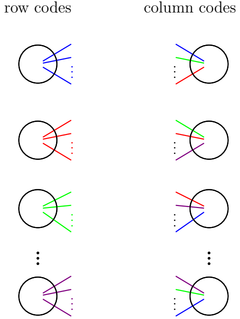

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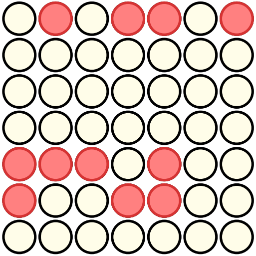
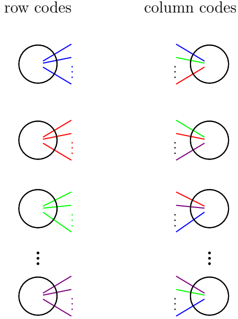


Column decoding



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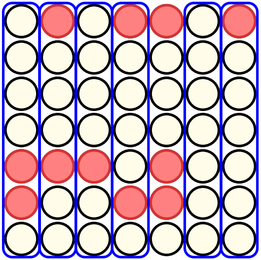

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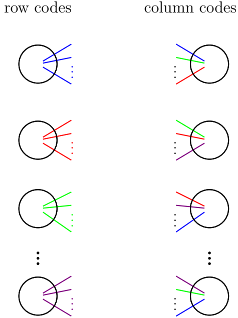

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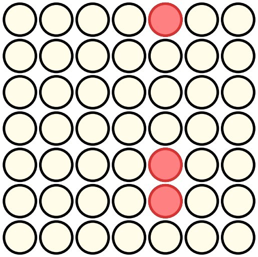

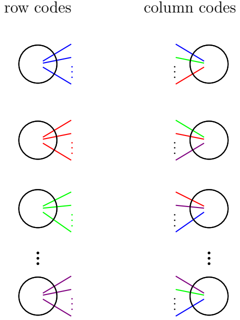

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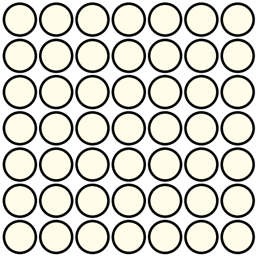
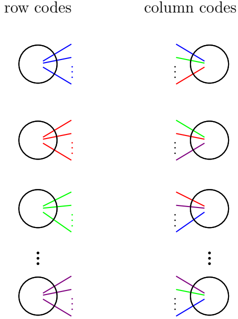


Decoding successful



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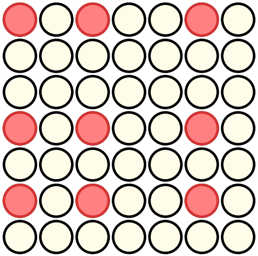
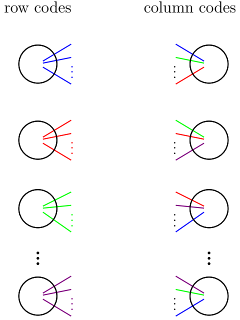


Or trapped in a stopping set



Density Evolution(DE) for Product Codes -Justesen et al

What is different about DE?

- Graph is highly structured
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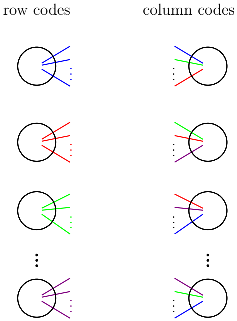
Main Idea

- Removal of corrected vertices (degree$\leq t$) from row codes $\Leftrightarrow$ removal of random edges from column codes uniformly at random
- # of errors in row/column changes after each iter
- Track the distribution



DE continued

Tail of the Poisson distribution

$$\pi_t(m) = \sum_{j \geq t} \mathrm{e}^{-m} m^j / j!$$

Effect of first step of decoding

If the # errors is Poisson with mean M, Mean # of errors after decoding is

$$m(1) = \sum_{j \geq t+1} j \mathrm{e}^{-M} M^j / j! = M \pi_t(M)$$

Evolution of degree distribution($d = 2$) - first iteration

Row decoding

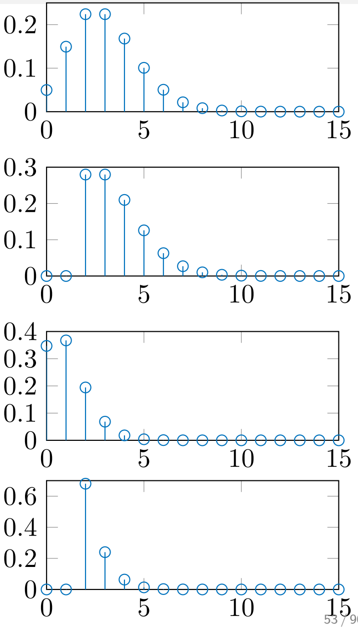
- Before row decoding
  - Distribution: Poisson(M), Mean: M
- After row decoding
  - Distribution: Truncated Poisson(M)
  - Mean: $M\pi_t(M) = m(1)$

Column decoding

- Before column decoding
  - Distribution: Poisson($m(1)$),Mean: $m(1)$
- After column decoding
  - Distribution: Truncated Poisson($m(1)$)
  - Mean: $m(2) = M\pi_t(m(1))$

After every decoding

- Distribution is a Truncated Poisson($m(j)$)
- $P[\#errors = i] = b\frac{m(j)^i}{i!}$



Evolution of the degree distribution - jth iteration

Recursion

- $m(0) = M$
- $m(1) = M\pi_t(M)$
- $m(j) = M\pi_t(m(j-1))$

Reduction in the parameter

- Average no. of errors in each row (column) $= m(j)\pi_t(m(j))$
- Decoding of rows reduces the parameter by $\frac{m(j)\pi_t(m(j))}{m(j-1)\pi_t(m(j-1))} = \frac{M\pi(m(j))}{m(j-1)}$
- New parameter is $m(j+1) = M\pi(m(j))$

Threshold

In the limit of large $\tilde{n}$ (length in each dimension), a t-error correcting product code can correct $\tilde{n}M$ errors when

$$M < \min_m \{\frac{m}{\pi_t(m)}\}$$

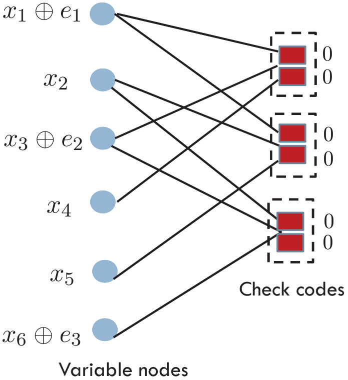
Thresholds for asymptotically large field size

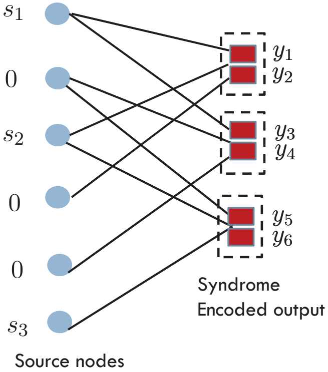
$$\text{Threshold} = \frac{\#\ of parity symbols}{\#\ of errors}$$

	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
$t=1$	4.0	2.4436	2.5897	2.8499	3.1393	3.4378	3.7383
$t=2$	2.3874	2.5759	2.9993	3.4549	3.9153	4.3736	4.8278
$t=3$	2.3304	2.7593	3.3133	3.8817	4.4483	5.0094	5.5641
$t=4$	2.3532	2.9125	3.5556	4.2043	4.8468	5.4802	6.1033

Notice that $L, K = O\left(N^{\frac{1-d}{d}}\right)$

Syndrome source coding





- $H\underline{x} = 0$
- Receive - $\underline{r} = \underline{x} \oplus \underline{e}$
- $H\underline{r} = H\underline{e} = \underline{y}$
- Recover $\underline{x}$ and sparse $\underline{e}$

- $H\underline{s} = \underline{y}$
- Set $\underline{r} = 0$ (Let a genie add $\underline{x}$ to $\underline{r}$)
- $\underline{y}$ is given to the decoder
- Recover sparse $\underline{s}$

Application 2

Sparse Fast Fourier Transform (SFFT) Computation

Problem Statement

$x[n]$: Time domain signal of length N whose spectrum is K-sparse

$$x[n] \xrightarrow{\text{DFT}} \underbrace{X[k]}_{(K-\text{sparse})}$$

Compute the locations and values of the K non-zero coefficients w.h.p

Sparse Fast Fourier Transform (SFFT) Computation

Problem Statement

$x[n]$: Time domain signal of length N whose spectrum is K-sparse

$$x[n] \xrightarrow{\text{DFT}} \underset{(K-\text{sparse})}{X[k]}$$

Compute the locations and values of the K non-zero coefficients w.h.p

Fast Fourier Transform (FFT)

- Sample complexity: N samples
- Computational complexity: $O(N \log N)$

We want sublinear sample and computational complexity

Sparse Fast Fourier Transform (SFFT) Computation

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Compute the locations and values of the K non-zero coefficients w.h.p

Related work

- Spectral estimation - Prony's method
- More recently Pawar and Ramchandran'13, Hassanieh, Indyk, Katabi'12

SFFT - A Sparse Graph Based Approach

Main Idea - Pawar and Ramchandran 2013

- Sub-sampling in time corresponds to aliasing in frequency
- Aliased coefficients $\Leftrightarrow$ parity check constraints of GLDPC codes
- CRT guided sub-sampling induces a code good for Peeling decoder
- Problem is identical to syndrome source coding

SFFT - A Sparse Graph Based Approach

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FFAST for Computing the DFT - Pawar and Ramchandran 2013

- Sampling complexity: $M = O(K)$ time domain samples
- Computational complexity: $O(K \log K)$

Subsampling and Aliasing - A Quick Review

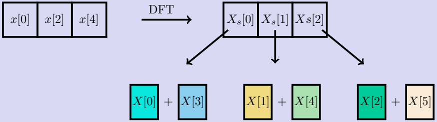
Subsampling results in aliasing

- Let $x[n] \xrightarrow{N-DFT} X[k], \quad k, n = 0, 1, \ldots, N-1$
- Let $x_s[n] = x[mL], \quad m = 0, 1, \ldots, N/L = M$ be a sub-sampled signal
- Let $x_s[m] \xrightarrow{M-DFT} X_s[l]$ be the DFT of the sub-sampled signal
- $\boxed{X_s[l] = M \sum_{p=0}^{L-1} X[l + pM]}$

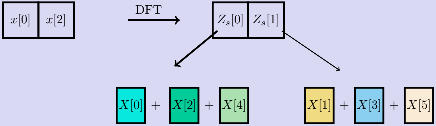
Aliasing and Sparse Graph Codes

$$\boxed{x[0]}\boxed{x[1]}\boxed{x[2]}\boxed{x[3]}\boxed{x[4]}\boxed{x[5]} \xrightarrow{\text{DFT}} \boxed{X[0]}\boxed{X[1]}\boxed{X[2]}\boxed{X[3]}\boxed{X[4]}\boxed{X[5]}$$

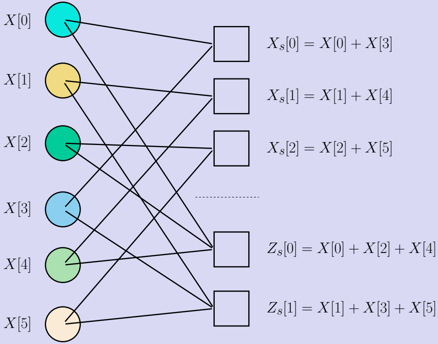
x_s: Sub-sampled by $f_1 = P_1 = 2$



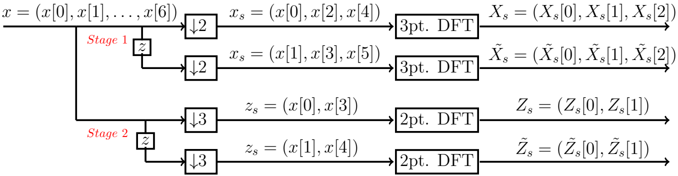
z_s: Sub-sampled by $f_2 = P_2 = 3$



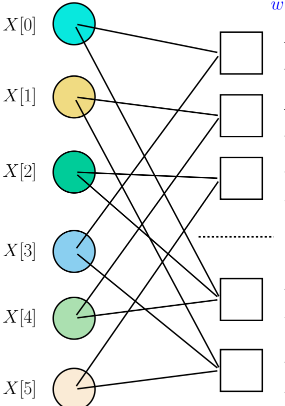
Factor graph



FFAST Algorithm Example



$w = e^{-j\frac{2\pi}{6}}$



$X_s[0] = X[0] + X[3]$
$\tilde{X}_s[0] = X[0]w^0 + X[3]w^3$

$X_s[1] = X[1] + X[4]$
$\tilde{X}_s[1] = X[1]w^1 + X[4]w^4$

$X_s[2] = X[2] + X[5]$
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$Z_s[0] = X[0] + X[2] + X[4]$
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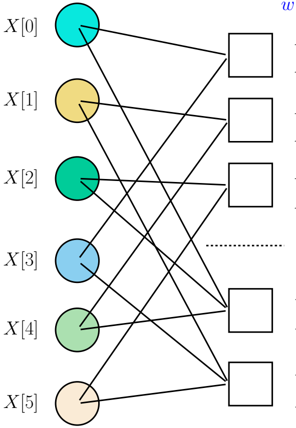
$\tilde{Z}_s[1] = X[1]w^1 + X[3]w^3 + X[5]w^5$

Singleton Detection



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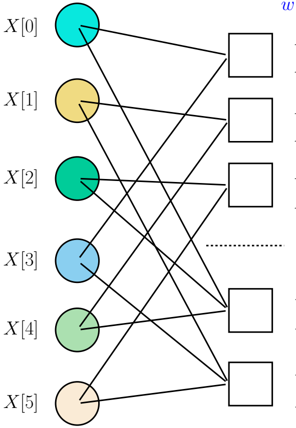
Singleton condition for a checknode

- Let $i = \frac{-N}{j2\pi}\log(\frac{\tilde{X}_s[l]}{X_s[l]})$. If $0 \leq i \leq N-1$, then checknode l is a Singleton.
- $Pos(l) = i$ is the only variable node participating and $X_s[l]$ is its value.

FFAST Decoder



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Peeling decoder

- 1 non-zero value among the neighbors of any right node can be recovered
- Iteratively errors can be corrected and analyzed for random non-zero coeffs

FFAST Decoder Example

Example 1

Let $N = 6$, and the non-zero coefficients be X[0]=5, X[3]=4, X[4]=7

FFAST Decoder Example

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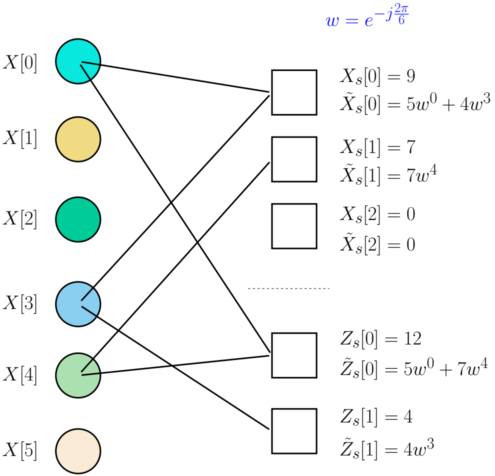

FFAST Decoder Example

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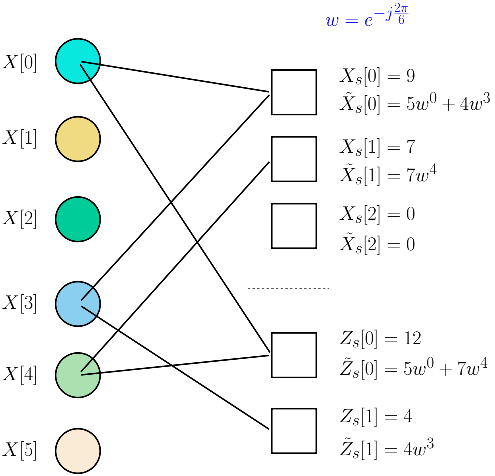
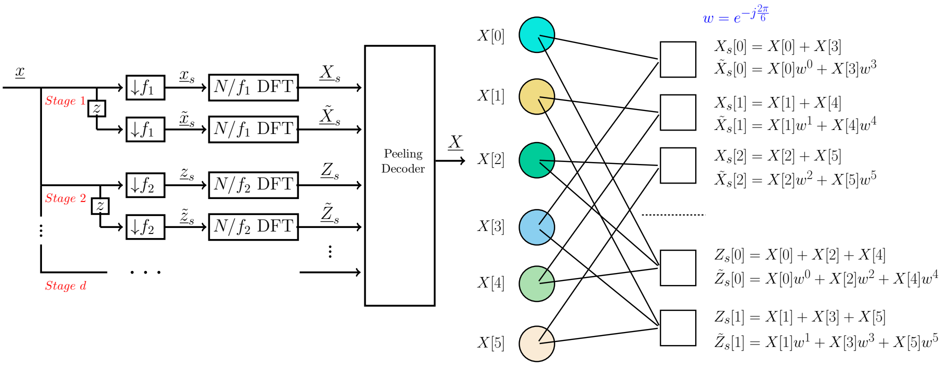


Yes, recoverable!

Generalization



Reed Solomon component codes

- $(X_s[l_1], \tilde{X}_s[l_1])$ correspond to 2 syndromes of a 1-error correcting RS code
- RS is over the complex field, no miscorrection

Product codes and FFAST ($d = 2$)

- X: K-sparse spectrum of length $N = P_1 P_2$ (P_1 and P_2 are co-prime)
- X': $P_1 \times P_2$ matrix formed by rearranging X according to mapping $\mathcal{M}$

$$
\begin{aligned}
X_s[l_1] &= \sum_{i=0}^{P_2-1} X[l_1 + iP_1], \quad 0 \le l_1 \le P_2 - 1 \\
Z_s[l_2] &= \sum_{i=0}^{P_1-1} X[l_2 + iP_2], \quad 0 \le l_2 \le P_1 - 1
\end{aligned}
$$

$$
\begin{array}{c|c|c|c|c|}
 & Z_s[0] & Z_s[1] & Z_s[2] & Z_s[3] \\
 & \downarrow & \downarrow & \downarrow & \downarrow \\
\hline
X_s[0] \longrightarrow & X[0] & X[5] & X[10] & X[15] \\
\hline
X_s[1] \longrightarrow & X[16] & X[1] & X[6] & X[11] \\
\hline
X_s[2] \longrightarrow & X[12] & X[17] & X[2] & X[7] \\
\hline
X_s[3] \longrightarrow & X[8] & X[13] & X[18] & X[3] \\
\hline
X_s[4] \longrightarrow & X[4] & X[9] & X[14] & X[19] \\
\hline
\end{array}
$$

Mapping

The mapping from $X(r)$ to $X'(i, j)$ is given by

$$(i, j) = \mathcal{M}(r) \equiv (r \mod P_2,\ r \mod P_1).$$

Note: CRT ensures that $\mathcal{M}$ is bijective

Product codes and FFAST ($d \geq 3$)

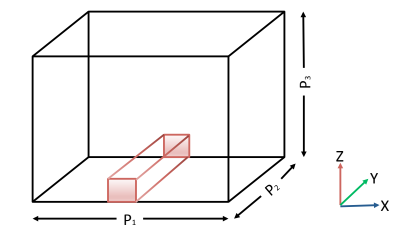
$N = P_1 \times P_2 \times \ldots \times P_d$

$$(i_1, i_2, \ldots, i_d) = \mathcal{M}(r) \equiv (r \mod f_1, r \mod f_2, \ldots, r \mod f_d).$$

Less-sparse regime

$f_i = N/P_i, \; i = 1, 2, \ldots, d$

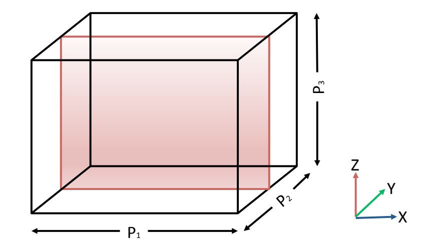
$\underline{d = 3}$



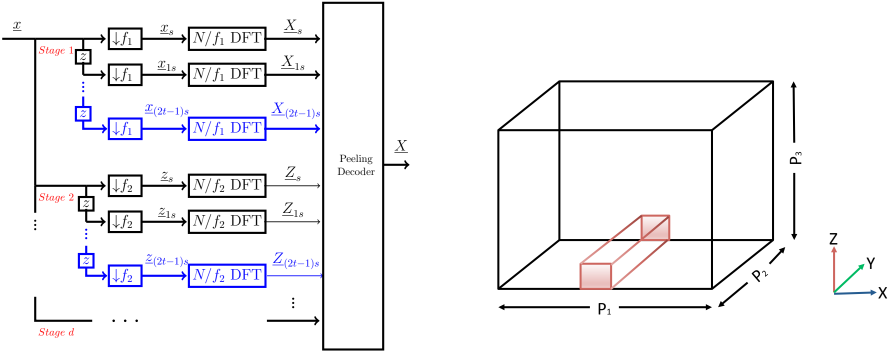
Very-sparse regime

$f_i = P_i, \; i = 1, 2, \ldots, d$

$\underline{d = 3}$



Connections between FFAST and Product Codes



FFAST	$\Leftrightarrow$	Product codes
d stages	$\Leftrightarrow$	d-dimensional product code
$2t$ branches	$\Leftrightarrow$	t-error correcting RS component codes
Non-zero coefficients	$\Leftrightarrow$	Error locations
Recovery of coefficients	$\Leftrightarrow$	Iterative decoding

Thresholds

Theorem 1

Less sparse case: In the limit of large P, the FFAST algorithm with d branches and $2t$ stages can recover the FFT coefficients w.h.p if $K < \frac{2dt}{c_{d,t}}$.

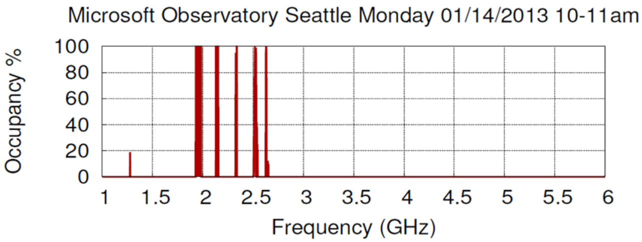
$$c_{d,t} = \min_m \{m/\pi^{d-1}(m)\}$$

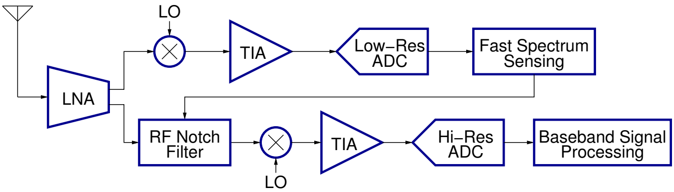
$$\text{Threshold} = \frac{\#\ of\ measurements}{recoverable\ sparsity} = \frac{2dt}{c_{d,t}}$$

	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
$t=1$	4.0	2.4436	2.5897	2.8499	3.1393	3.4378	3.7383
$t=2$	2.3874	2.5759	2.9993	3.4549	3.9153	4.3736	4.8278
$t=3$	2.3304	2.7593	3.3133	3.8817	4.4483	5.0094	5.5641
$t=4$	2.3532	2.9125	3.5556	4.2043	4.8468	5.4802	6.1033

Notice that $L, K = O\left(N^{\frac{1-d}{d}}\right)$

Interference-tolerant A/D Converter

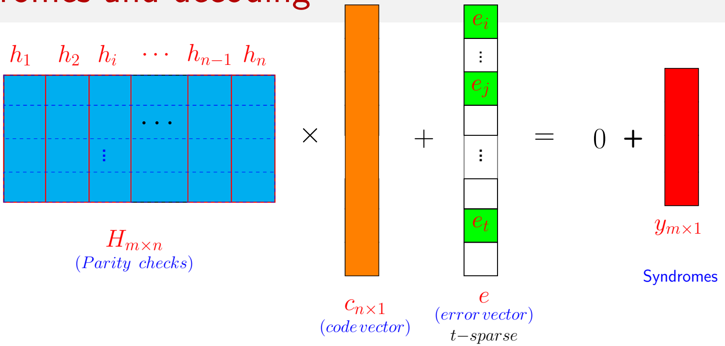




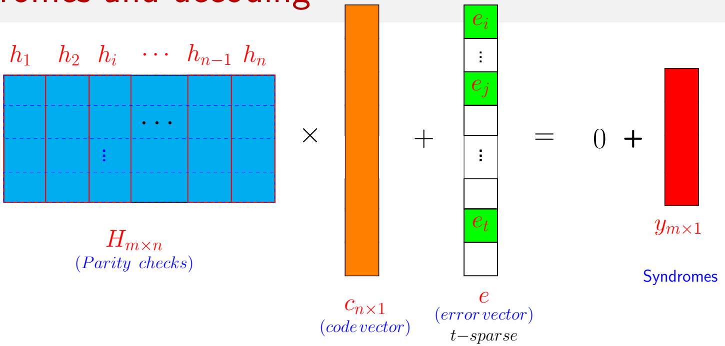
Open problems

- If we use MAP decoding, is the subsampling procedure optimal?
- What happens when $N = 2^i$?
- Bursty case? Can we have threshold theorems?
- Using this idea in actual applications

Syndromes and decoding

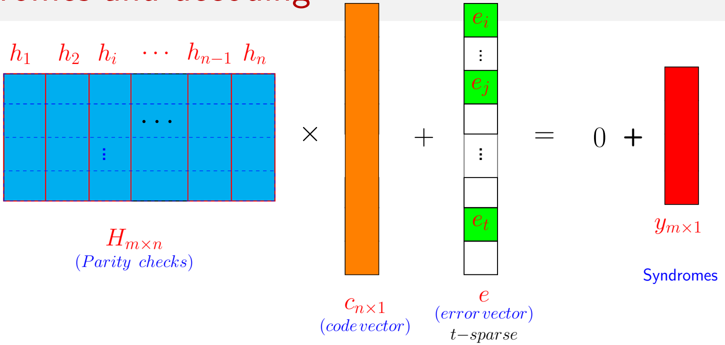


Syndromes and decoding



- Syndrome : Linear combination of $\underline{h}_i$s, i.e., $\underline{y} = e_i\underline{h}_i \oplus e_j\underline{h}_j \oplus e_t\underline{h}_t$
- Decoding : Find min weight $\underline{e}$: $\underline{y} = e_i\underline{h}_i \oplus e_j\underline{h}_j \oplus e_t\underline{h}_t$

Syndromes and decoding



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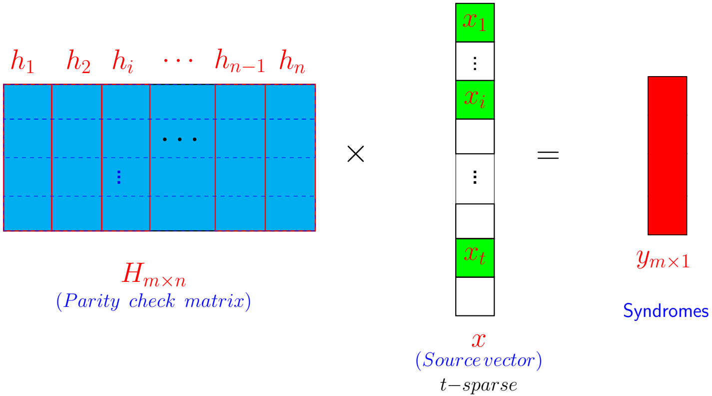
Coding theory deals with the construction of $\mathbf{H}$ and efficient decoding algorithms, i.e., given a linear combination of the columns of $\mathbf{H}$, it develops tools to determine a sparse $\underline{e}$

Syndrome source coding



Compression of a sparse binary source

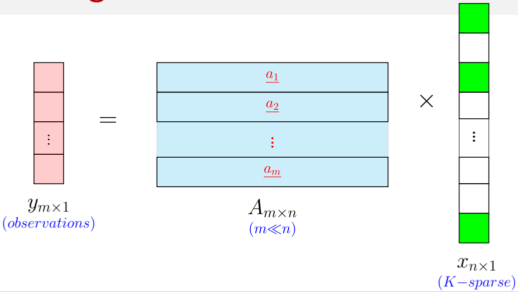
- Compressed version is the syndrome $\underline{y}$
- Reconstruction is the same as decoding
- Similar to the canonical sparse recovery problem

Review of primitives

- Idea of a check node or a measurement node which is a function of some symbols
- Singleton detection - be able to identify one non-zero symbol
- Peeling - if we know some symbols, be able to remove and adjust measurement

Application 3

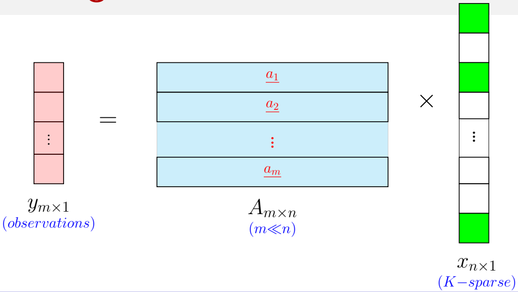
Compressed sensing



Classical compressed sensing

- $\underline{x}$ is a K-sparse vector over $\mathbb{R}$ or $\mathbb{C}$
- We 'compress' $\underline{x}$ by storing only $\underline{y} = \mathbf{A}\ \underline{x}$
- Reconstruction - Solve $\underline{\hat{x}} = \arg\min ||\underline{z}||_0 : \underline{y} = \mathbf{A}\underline{z}$
- CS - Solve $\underline{\hat{x}} = \arg\min ||\underline{z}||_1 : \underline{y} = \mathbf{A}\underline{z}$

Compressed sensing



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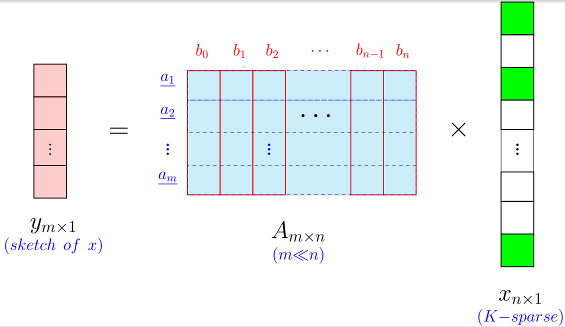
Coding theoretic approach - syndrome source coding over complex numbers

- Sensing matrix $\mathbf{A} \Leftrightarrow$ Parity check matrix $\mathbf{H}$

Data stream computing

Problem - consider a router in a large network

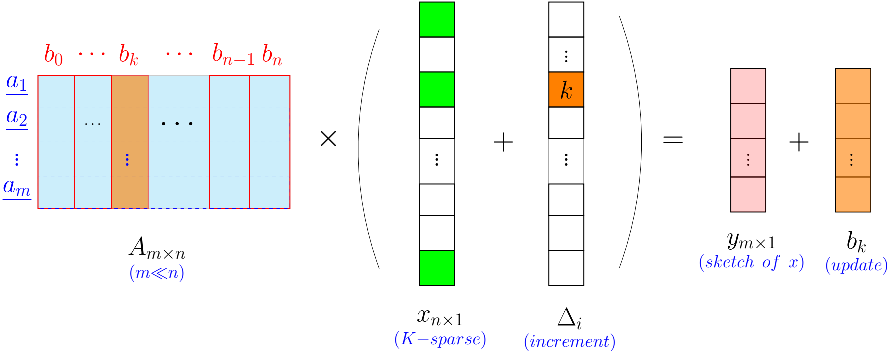
- Count the number of packets from source i to destination j, say x_{ij}
- Data vector is huge, $n = 2^{64}$
- Heavy hitters - only a few of them are large



Keep only a low dimensional ($m \ll n$) sketch of x

- $\underline{y}_{m\times 1} = \mathbf{A}\underline{x} \Leftrightarrow$ Syndrome, $x_{i,j} \in \mathbb{Z}^+$
- Reconstruction is same as decoding

Incremental updates

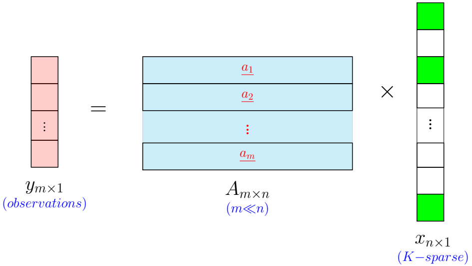


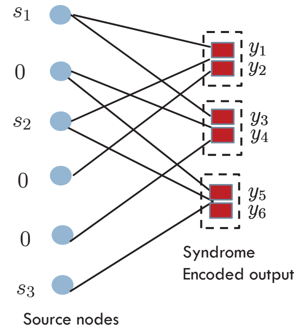
Sketch y supports incremental updates to x as the sketching procedure is linear.

$$x + \Delta_i \to y + A\Delta_i$$

(adding ith column vector of A to existing sketch)

Compressed Sensing (Li, Ramchandran '14)





Sketching matrix ($A_{m\times n}$)

$$A_{m\times n} = \underset{(d-left\ regular\ Graph)}{\mathbf{H}_{\frac{\mathbf{m}}{\mathbf{2}}\times \mathbf{n}}} \otimes \underset{(Singleton\ identifier)}{\mathbf{B}_{\mathbf{2}\times \mathbf{n}}}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & \cdots & W^{n-1} \end{bmatrix} \quad W = e^{j\frac{2\pi}{n}}$$

Main results for compressed sensing

Noiseless case

- Samples : $2K$ versus Info-theoretic limit $K+1$
- Computations: $O(K)$ versus $O(K^2)$
- If $K = O(n^\delta)$, small price to pay in terms of samples

Noisy case

- Sample: $O(k \log^{1.3\cdot} n)$ vs limit: $O(k \log(n/k))$ necessary and sufficient
- Computations: $O(k \log^{1.3\cdot} n)$

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Vem, Thenkarai Janakiraman, N. ITW'16

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- Computations: $O(k \log^{1.3\cdot}(n/k))$

Group Testing (Lee, Pedarsani, Ramchandran '15)

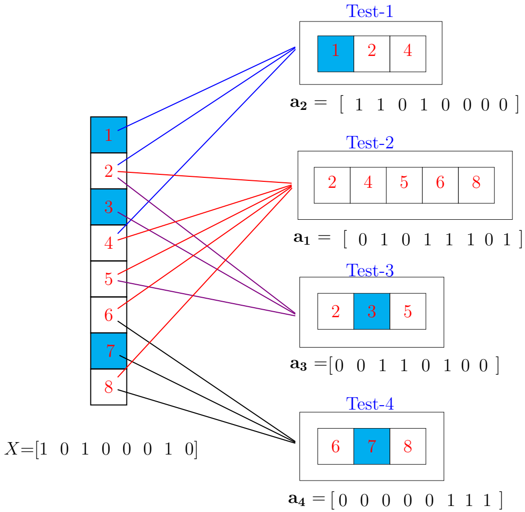
- II World War - detect all soldiers with syphilis
- Tests performed on efficiently pooled groups of items
- Least no. of tests (m) to identify k defective items from n items

Group Testing

Example

Test Results (Observations)

$$\boxed{y_i = \bigvee_{j=1}^{N} a_{ij} X_j = < a_i, X >}$$



Positive

$y_1 = 1$

Negative

$y_1 = 0$

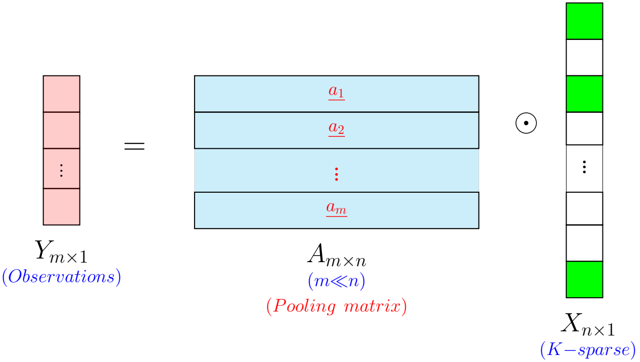
Positive

$y_1 = 1$

Positive

$y_1 = 1$

Group Testing



$$\underset{(Observation\ vector)}{Y_{m\times 1}} = A \odot X = \begin{bmatrix} < a_1, X > \\ < a_2, X > \\ \vdots \\ < a_m, X > \end{bmatrix} \qquad < a_i, X > = \bigvee_{j=1}^{N} a_{ij} X_j$$

Group Testing

Singleton detection

$$\begin{bmatrix} \frac{H_1}{H_1} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{b_1}}{\mathbf{b_1}} & \frac{\mathbf{b_2}}{\mathbf{b_2}} & \frac{\mathbf{b_3}}{\mathbf{b_3}} & \cdots & \frac{\mathbf{b_{n-1}}}{\mathbf{b_{n-1}}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 \\ -- & -- & -- & -- & -- & -- \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

Note: If a checknode is a singleton, with ith bit-node participating, then the observation vector is the ith column of A.

- Singleton - if the weight of first two observation vectors together is L.
- Position of the defective item is - decimal value of the 1st observation vector.

Group Testing

Measurement matrix ($A_{m\times n}$)

$$A_{m\times n} = \underbrace{\mathbf{G}_{\frac{m}{6}\times n}}_{(d\text{-}left\ regular\ Graph)} \otimes \underbrace{\mathbf{H}_{6\times n}}_{(Singleton\ identifier)}$$

Let, $\mathbf{b_i}$ denote the L-bits binary representation of the integer $i-1$, $L = \lceil \log_2 n \rceil$.

$$H = \begin{bmatrix} \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} & \cdots & \mathbf{b_{n-1}} \\ \overline{\mathbf{b_1}} & \overline{\mathbf{b_2}} & \overline{\mathbf{b_3}} & \cdots & \overline{\mathbf{b_{n-1}}} \\ \mathbf{b_{i_1}} & \mathbf{b_{i_2}} & \mathbf{b_{i_3}} & \cdots & \mathbf{b_{i_{n-1}}} \\ \overline{\mathbf{b_{i_1}}} & \overline{\mathbf{b_{i_2}}} & \overline{\mathbf{b_{i_3}}} & \cdots & \overline{\mathbf{b_{i_{n-1}}}} \\ \mathbf{b_{j_1}} & \mathbf{b_{j_2}} & \mathbf{b_{j_3}} & \cdots & \mathbf{b_{j_{n-1}}} \\ \overline{\mathbf{b_{j_1}}} & \overline{\mathbf{b_{j_2}}} & \overline{\mathbf{b_{j_3}}} & \cdots & \overline{\mathbf{b_{j_{n-1}}}} \end{bmatrix}$$

$s_1 = (i_1, i_2, \cdots, i_{n-1})$ and $s_2 = (j_1, j_2, \cdots, j_{n-1})$ are permutations

Decoding procedure

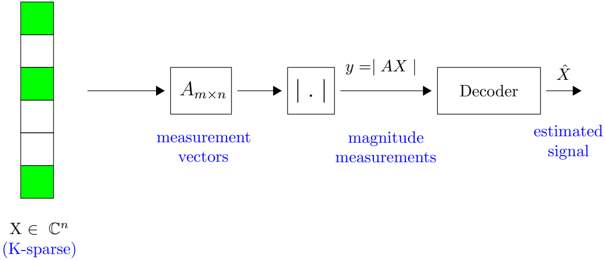
- Identify and decodes singletons using weights of the observation vector
- Identify and resolve doubletons by guessing to satisfy the first pair of observation vectors and checking if the guess satisfies the other two pairs of observations

Main results for group testing

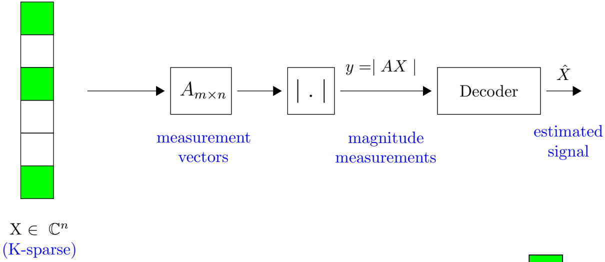
Non-adaptive Group Testing (Noiseless and Noisy)

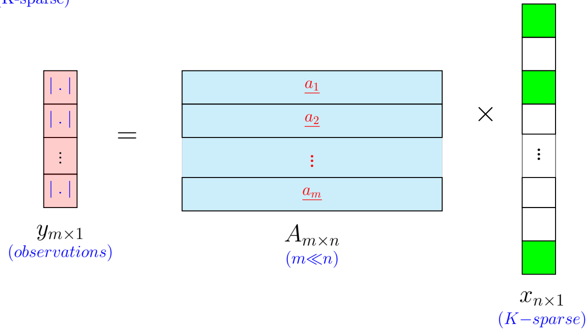
- Recovers $(1-\epsilon)k$ items with h.p.
- Samples: $m = O(k \log_2 n)$ versus limit: $\Theta(k \log(\frac{n}{k}))$
- Computational complexity: $O(k \log n)$ (order optimal)

Compressive Phase Retrieval



Compressive Phase Retrieval





Conclusion

- Review of a simple message passing decoder called the peeling decoder
- Density evolution as a tool to analyze its asymptotic performance
- Applications
  - Massive uncoordinated multiple access
  - Sparse Fourier transform computation
  - Compressed sensing type sparse recovery problems

Questions?

Thank you!