Upper Bounds on the MAP Threshold of Iterative Decoding Systems with Erasure Noise

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***Abstract*—Following the work of Méasson, Montanari, and Urbanke, this paper considers the maximum a *posteriori* (MAP) decoding thresholds of three iterative decoding systems. First, irregular repeat-accumulate (IRA) and accumulate-repeat-accumulate (ARA) code ensembles are analyzed on the binary erasure channel (BEC). Next, the joint iterative decoding of LDPC codes is studied on the dicode erasure channel (DEC). The DEC is a two-state intersymbol-interference (ISI) channel with erasure noise, and it is the simplest example of an ISI channel with erasure noise. The MAP threshold bound for the joint decoder is based on a slight generalization of the EXIT area theorem.**

***Index Terms*—MAP threshold, iterative decoding, LDPC codes, EXIT function, erasure channel.**

I. Introduction

A thorough analysis of iterative decoding systems and the relationship between maximum *a posteriori* (MAP) and belief propagation (BP) decoding was initiated by Méasson, Montanari, and Urbanke in [1], [2]. This analysis is based on density evolution (DE) and extrinsic information transfer (EXIT) functions [3]. Their work focuses mainly on low-density parity-check (LDPC) and turbo codes, but they note that these ideas can be extended to other iterative decoding systems. In this paper, we extend some of their results to irregular repeat-accumulate (IRA), accumulate-repeat-accumulate (ARA), and the joint iterative decoding of LDPC codes over channels with memory.

DE is a method of evaluating iterative decoding systems for asymptotically large block lengths and was introduced in [4]. EXIT functions were introduced by ten Brink as an approximate technique to visualize the convergence of iterative systems [3]. In fact, for the erasure channel, EXIT functions satisfy a rigorous conservation law known as the area theorem [5]. The area theorem can be used to rigorously connect the performance of a code under MAP decoding to its performance under BP decoding. Méasson, Montanari and Urbanke give a graphical construction of the MAP threshold using an approach reminiscent of the Maxwell construction in thermodynamics to provide a bridge between MAP and BP decoding [1], [2].

Jin, Khandekar, and McEliece proposed and analyzed IRA codes in [6]. ARA codes were introduced by Abbasfar, Divsalar, and Kung in [7]. Later, it was shown that the DE analysis of IRA and ARA codes can be reduced to the DE analysis of LDPC codes via a technique known as graph reduction [8].

The idea of decoding a code transmitted over a channel with memory via iteration was first introduced by Douillard, et al. in the context of turbo codes and is known as *turbo equalization* [9]. Turbo equalization can also be extended to the joint decoding of LDPC codes by constructing one large graph which represents the constraints of both the channel and the code [10]. For finite-state (FS) channels, analysis of joint decoding requires the analysis of the BCJR algorithm which is used to decode the channel. For some channels, DE can be done analytically for the joint iterative decoding of irregular LDPC codes and the channel [11]. One such channel is the dicode erasure channel (DEC), which is simply a binary-input channel with a linear response of $1-D$ and erasure noise.

In this paper, we apply the ideas of [12], [1], [2] to IRA ensembles, ARA ensembles, and the joint iterative decoding of irregular LDPC codes on the DEC. Both the MAP and BP erasure thresholds are computed and compared with each other.

In Section II, a brief background is given for iterative decoding, DE, EXIT functions, and the MAP threshold bounding technique. In Section III, the MAP threshold bounding technique is applied to IRA and ARA codes. In Section IV, joint iterative decoding is briefly introduced and the MAP threshold bounding technique is applied to joint decoding. Finally, concluding remarks and open questions are discussed in Section V.

II. Background

A. Low-Density Parity-Check Codes

Low-density parity-check (LDPC) codes are linear codes which have a sparse graph representation; in general, they tend to exhibit good performance under message-passing decoding. A (d_v, d_c)-regular LDPC code is a binary linear code such that every bit node has degree d_v and every check node has degree d_c. An irregular LDPC ensemble is described by its degree distribution (d.d.), which encodes the fraction of nodes (or edges) with a particular degree. From an edge perspective, the d.d. of the bit and check nodes is given, respectively, by $\lambda(x) = \sum_{i=1}^{\infty} \lambda_i x^{i-1}$ and $\rho(x) = \sum_{i=1}^{\infty} \rho_i x^{i-1}$, where λ_i (or ρ_i) represent the fraction of edges attached to a bit node (or check node) of degree i. From a node perspective, the d.d. of bit and check nodes is given, respectively, by $L(x) = \sum_{i=1}^{\infty} L_i x^i$ and $R(x) = \sum_{i=1}^{\infty} R_i x^i$, where L_i (or R_i) represent fraction of bit (or check) nodes of degree i. Codes are chosen randomly from an ensemble by choosing a random permutation to connect the bit and check nodes [13, p. 579][14]. The design rate of the code in terms of its degree distribution is given by

$$r_{LDPC} = 1 - \frac{L'(1)}{R'(1)} = 1 - \frac{\int_0^1 \rho(x)\,dx}{\int_0^1 \lambda(x)\,dx}.$$

The performance of irregular LDPC codes can be significantly better than regular LDPC codes. Certain structural modifications, such as those provided by IRA and ARA constructions can also improve performance. DE can be used to analyze and design (e.g., optimize the degree distribution) LDPC, IRA, and ARA codes. DE works by recursively tracking the distribution of messages passed around the Gallager-Tanner-Wiberg (GTW) graph during iterative decoding. It also gives a precise characterization of the asymptotic performance in terms of a noise threshold, where decoding almost surely converges if the noise is less than the threshold. For a BEC(ϵ) (i.e., a binary erasure channel with erasure probability ϵ), the DE recursion can be written in closed form as $x_{i+1} = \epsilon\lambda(1-\rho(1-x_i))$, where x_i is the average fraction of erasure messages sent from the bit nodes to the check nodes during iteration i.

B. EXIT Functions and the Area Theorem

EXIT functions first appeared as handy tools to visualize the iterative decoding process; from EXIT curves, one can easily see the "bottlenecks" in the iterative decoding process [3]. Once these critical regions have been identified, the component codes can be changed appropriately to "match" the curves and improve the performance of the system.

Definition 1: [1], [2] Let $\mathcal{C}$ be a length-n binary code defined by the probability distribution $p_{X_1^n}(x_1^n)$. Let X_1^n be chosen according to $p_{X_1^n}(x_1^n)$ and Y_1^n be the result of transmitting X_1^n over a BEC(ϵ). Then, the *MAP EXIT function* is defined to be

$$h^{MAP}(\epsilon) \triangleq \frac{1}{n}\sum_{i=1}^{n} H\left(X_i | Y_1^n(\epsilon) \backslash Y_i(\epsilon)\right).$$

Remark 1: From this, we see that $h^{MAP}(\epsilon)$ is the average (over all bits) entropy of the optimal *a posteriori* probability (APP) estimate of X_i from the observations Y_1^n except Y_i. The notation $Y_1^n(\epsilon)$ and $Y_i(\epsilon)$ is used to emphasize the dependence of these r.v. on ϵ. Let ϵ^{MAP} be the erasure threshold of MAP decoding for a code ensemble. For asymptotically large n, the average conditional entropy $h^{MAP}(\epsilon)$ converges to zero for $\epsilon < \epsilon^{MAP}$ and is strictly positive for $\epsilon > \epsilon^{MAP}$.

Theorem 1 (Area Theorem): Let $\mathcal{C}$ be a length-n binary code defined by the probability distribution $p_{X_1^n}(x_1^n)$. Let X_1^n be chosen according to $p_{X_1^n}(x_1^n)$ and Y_1^n be the result of transmitting X_1^n over a BEC(ϵ). To emphasize that Y_1^n depends on the channel parameter δ write $Y_1^n(\delta)$. Then

$$\frac{1}{n}H\left(X_1^n | Y_1^n(\delta)\right) = \int_0^{\delta} h^{MAP}(\epsilon)\,d\epsilon.$$

Proof: A nice history of this theorem and its various proofs can be found in [2, p. 44]. ■

In addition, there is another, perhaps more surprising, application of EXIT functions; they can be used to connect the performance of a code under BP decoding to that under MAP decoding.

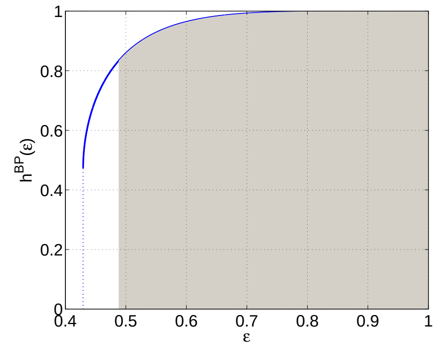

Figure 1. The BP EXIT function $h^{BP}(\epsilon)$ of a (3,6)-regular LDPC code on the erasure channel. The left boundary of the shaded area is the upper bound on ϵ^{MAP}, and the area of the shaded portion under the curve equals the code rate of $\frac{1}{2}$. This gives $\epsilon^{BP} = 0.4294$ and $\epsilon^{MAP} \leq 0.4881$.

Definition 2: The *BP EXIT function* of a length-n code is given by

$$h^{BP}(\epsilon) \triangleq \frac{1}{n}\sum_{i=1}^{n} h_i^{BP}(\epsilon),$$

where $h_i^{BP}(\epsilon)$ is the entropy of the iterative decoding estimate of X_i from Y_1^n except Y_i. The iterative decoding estimate of a bit is given by the bit's extrinsic message in the BP decoder after l iterations of decoding.

Remark 2: For ensembles of codes, these expressions will also refer to the asymptotic EXIT functions as $n \to \infty$. In the case of BP EXIT functions, we assume also that the number of decoding iterations $l \to \infty$ as well (with the n-limit taken first). These limits are well-defined and deterministic for BP EXIT functions because of the concentration theorem [14]. For the MAP EXIT function, a similar approach can be used to show that $h^{MAP}(\epsilon)$ concentrates around its ensemble average [15] which we will assume to be well-defined.

The following parametric expression for the asymptotic BP EXIT function is given in [1], [2] for the standard ensemble of LDPC codes.

Theorem 2: For an irregular LDPC code, the asymptotic BP EXIT curve is given in parametric form by

$$h^{BP}(\epsilon(x)) = \begin{cases} 0, & x \in [0, x^{BP}) \\ L(1-\rho(1-x)) & x \in (x^{BP}, 1] \end{cases},$$

where $\epsilon(x) \triangleq \frac{x}{\lambda(1-\rho(1-x))}$, and x^{BP} denotes the location of the unique minimum of $\epsilon(x)$ in the range (0,1] and $\epsilon^{BP} \triangleq \epsilon(x^{BP})$ is the BP decoding threshold.

The BP EXIT function $h^{BP}(\epsilon)$ of a (3,6)-regular LDPC code on the erasure channel is shown in Fig 1. Its BP threshold ϵ^{BP} is given by the point where $h^{BP}(\epsilon)$ drops down to zero.

C. Bounding the MAP Decoding Threshold

The following approach to bounding the MAP decoding threshold is based on the approach used in [1], [2]. The key point is that the optimality of the MAP decoder implies $h^{MAP}(\epsilon) \leq h^{BP}(\epsilon)$. Since the integral of $h^{MAP}(\epsilon)$ is equal

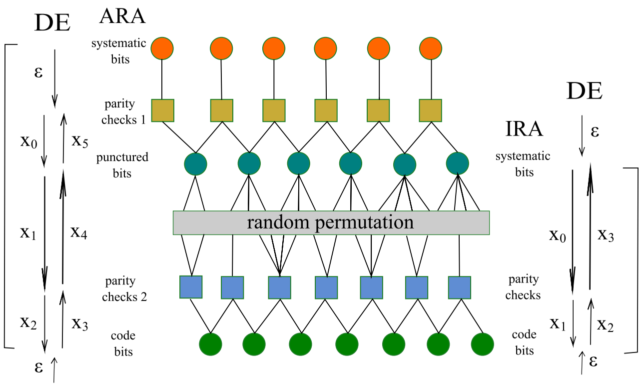

Figure 2. Gallager-Tanner-Wiberg graph for ARA and IRA codes

to the code's true rate r (based on the area theorem), it follows that

$$r_{LDPC} \leq r = \int_{\epsilon^{MAP}}^{1} h^{MAP}(\epsilon)\, d\epsilon \leq \int_{\epsilon^{MAP}}^{1} h^{BP}(\epsilon)\, d\epsilon$$

because $h^{MAP}(\epsilon) = 0$ for $0 \leq \epsilon \leq \epsilon_{MAP}$ and $r_{LDPC} \leq r$ (i.e., linear dependencies in the parity-check matrix can only increase the rate). This bound is useful because $h^{BP}(\epsilon)$ can be computed easily. In some cases, it can also be shown that the bound is tight and that $h^{MAP}(\epsilon) = h^{BP}(\epsilon)$ for $\epsilon > \epsilon^{MAP}$ [1], [2].

Fig. 1 shows the BP EXIT function $h^{BP}(\epsilon)$ and the integral bound on ϵ^{MAP}. In this construction, the left edge of the shading is chosen so that the shaded area under the BP EXIT function curve equals the code rate. This left edge provides the upper bound on the MAP threshold ϵ^{MAP}.

III. MAP Threshold Bounds for IRA and ARA Codes

A. Background on IRA and ARA Codes

IRA and ARA codes can be viewed as subclasses of LDPC codes that have natural linear-time encoding algorithms [6], [7]. Using iterative sum-product decoding, they can also be decoded with a per-iteration complexity that is linear in the block length. From an encoding point of view, it is natural to view IRA and ARA codes as interleaved serially concatenated codes [8]. From a decoding point of view, they are easily seen to be sparse-graph codes compatible with belief propagation decoding. There are a few slightly different definitions of ARA ensemble, and this paper uses the ensemble and DE equations defined in [8].

B. MAP Threshold Bounds for systematic IRA Codes

1) Density Evolution and Fixed Point Analysis of Iterative Decoding for IRA codes: Since IRA codes can be viewed as LDPC codes with an accumulate structure attached to the check nodes (see Fig. 2), they can also be defined by their d.d. pair $\lambda(x), \rho(x)$. For any fixed number of decoding iterations l, the DE equations give (almost surely as $n \to \infty$) the erasure rate of the internal messages passed by the BP decoder for a random code and channel erasure pattern. In [6], for any fixed ϵ, the DE equations are given by

$$\begin{aligned}
x_0^{(l)} &= \epsilon\lambda\left(x_3^{(l)}\right)\\
x_1^{(l)} &= 1-\left(1-x_2^{(l-1)}\right)R\left(1-x_0^{(l-1)}\right)\\
x_2^{(l)} &= \epsilon x_1^{(l)}\\
x_3^{(l)} &= 1-\left(1-x_2^{(l)}\right)^2\rho\left(1-x_0^{(l-1)}\right),
\end{aligned}$$

where ϵ is the channel erasure probability and $x_i^{(l)}$ tracks the average fraction of erasure messages for edge type-i and iteration l. The rate r_{IRA} of a systematic IRA code given can be written as

$$r_{IRA} = \left(1 + \frac{\int_0^1 \rho(x)dx}{\int_0^1 \lambda(x)dx}\right)^{-1}.$$

2) BP EXIT Function and Bounds on the MAP Threshold:

Lemma 1: The asymptotic BP EXIT function of the IRA code ensemble is given by

$$h^{BP-IRA}(\epsilon) = r_{IRA}\, L(x_3) + (1-r_{IRA})\, x_1^2,$$

where x_0, x_3 are given by the $l \to \infty$ DE fixed point for ϵ.

Proof: IRA codes have multiple types of bits in the GTW graph. The $n(1-r)$ parity bits have an average extrinsic erasure probability of $(x_1^{(l)})^2$ after l iterations. Likewise, the nrL_d information bits of degree-d have an average extrinsic erasure probability of $(x_3^{(l)})^d$ after l iterations. Therefore, we can write the large-iteration long-block limit of the IRA code EXIT function as

$$\begin{aligned}
h^{BP-IRA}(\epsilon) &= \lim_{l\to\infty}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} h_i^{BP-IRA}(\epsilon)\\
&\overset{a.s.}{=} \lim_{l\to\infty}\left[r_{IRA}\sum_{d=1}^{\infty} L_d\left(x_3^{(l)}\right)^d + (1-r_{IRA})\left(x_1^{(l)}\right)^2\right]\\
&= r_{IRA}\, L(x_3) + (1-r_{IRA})\,(x_1)^2.
\end{aligned}$$

■

Accordingly, we plot the BP EXIT function of the IRA code and integrate backwards from the right end of the curve where $\epsilon = 1$. The integration process stops at ϵ^* when $\int_{\epsilon^*}^{1} h^{BP}(\epsilon)\, d\epsilon = r_{IRA}$. This gives the upper bound $\epsilon^{MAP} \leq \epsilon^*$ for the IRA code ensemble (Fig. 3).

C. MAP Threshold Bounds for ARA Codes

1) Density Evolution of Systematic ARA Ensembles: Pfister and Sason [8] consider the asymptotic analysis of ensembles of ARA codes under the assumption that the codes are transmitted over a BEC and decoded with an iterative messaging-passing decoder. For this ensemble, they find that DE for the BEC can be computed in closed form. From Fig. 2, we see that

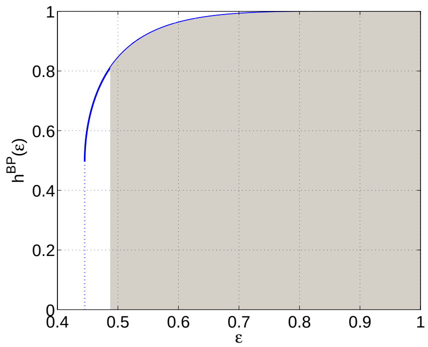

Figure 3. The BP EXIT function $h^{BP}(\epsilon)$ of a (4, 4)-regular IRA code (i.e., $\lambda(x) = \rho(x) = x^3$) on the erasure channel. The left boundary of the shaded area is the upper bound on ϵ^{MAP}, and the area of the shaded portion under the curve equals the code rate of $\frac{1}{2}$. This gives $\epsilon^{BP} = 0.4451$ and $\epsilon^{MAP} \leq 0.4872$.

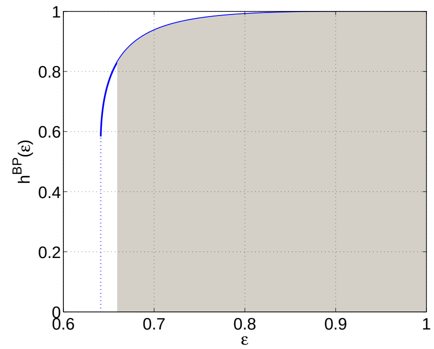

Figure 4. The BP EXIT function $h^{BP}(\epsilon)$ of the ARA code with $\lambda(x) = x^2$, $\rho(x) = \frac{2}{3}x + \frac{1}{3}$ on the erasure channel. The left boundary of the shaded area is the upper bound on ϵ^{MAP}, and the area of the shaded portion under the curve equals the code rate of $\frac{1}{3}$. This gives $\epsilon^{BP} = 0.6412$ and $\epsilon^{MAP} \leq 0.6593$.

$$
\begin{aligned}
x_0^{(l)} &= 1 - \left(1 - x_5^{(l-1)}\right)(1-\epsilon) \\
x_1^{(l)} &= \left(x_0^{(l)}\right)^2 \lambda\left(x_4^{(l-1)}\right) \\
x_2^{(l)} &= 1 - R\left(1 - x_1^{(l)}\right)\left(1 - x_3^{(l-1)}\right) \\
x_3^{(l)} &= \epsilon x_2^{(l)} \\
x_4^{(l)} &= 1 - \left(1 - x_3^{(l)}\right)^2 \rho\left(1 - x_1^{(l)}\right) \\
x_5^{(l)} &= x_0^{(l)} L\left(x_4^{(l)}\right),
\end{aligned}
$$

where ϵ is the channel erasure probability and $x_i^{(l)}$ tracks the average fraction of erasure messages for edge type-i and iteration l. The rate r_{ARA} of a systematic ARA code given can be written as

$$r_{ARA} = \frac{1}{1 + \frac{L'(1)}{R'(1)}}.$$

2) BP EXIT Function and Bounds on the MAP Threshold:

Lemma 2: The asymptotic BP EXIT function of the ARA code ensemble is given by

$$h^{BP-ARA}(\epsilon) = r_{ARA}\left[1 - (1-x_5)^2\right] + (1 - r_{ARA})\, x_2^2,$$

where x_5, x_2 are given by the $l \to \infty$ DE fixed point for ϵ.

Proof: ARA codes have two classes of bits are transmitted across the channel. The nr systematic bits have an average extrinsic erasure probability of $1-(1-x_5^{(l)})^2$ after l iterations. Likewise, the $n(1-r)$ code bits have an average extrinsic erasure probability of $(x_2^{(l)})^2$, after l iterations. Thus, we can write the large-iteration long-block limit of the ARA code EXIT function as

$$
\begin{aligned}
h^{BP-ARA}(\epsilon) &= \lim_{l\to\infty}\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} h_i^{BP-ARA}(\epsilon) \\
&= r_{ARA}\left[1 - (1-x_5)^2\right] + (1 - r_{ARA})\, x_2^2.
\end{aligned}
$$

■

The same integration process, that was used for LDPC and IRA codes, is used to calculate the upper bound on ϵ^{MAP} for ARA codes (Fig. 4).

D. Tightness of the Upper Bound

It is worth noting that, for the BEC, graph reduction can be used to reduce any IRA or ARA code into an LDPC code [8]. After this reduction, the LDPC code can be decoded with a peeling decoder until the decoder gets stuck. If one analyzes this process carefully, one can compute the d.d. of residual graph and apply the the counting argument of [1], [2] to (possibly) prove the tightness of the MAP threshold. We are currently pursuing this approach, but it is complicated by the fact that the residual graph has nodes of unbounded degree.

IV. MAP Threshold Bounds for Joint Decoding

A. Background and System Description

Pfister and Siegel consider the achievable rate of joint iterative decoding of LDPC codes and channels with memory [11]. Here we use same system model and consider instead the MAP decoding threshold. The block diagram of the system is shown in Fig. 5. It is a relatively standard setup for the joint iterative decoding of an LDPC code and a channel with memory. Equiprobable information bits, $U_1^k \in \{0,1\}^k$, are encoded into an LDPC codeword, $X_1^n \in \{0,1\}^n$, which is observed through the dicode erasure channel (DEC) as the output vector, $Y_1^n \in \{-1,0,1,?\}$. The decoder consists of the channel APP detector an LDPC decoder which pass messages back and forth. In the first half of decoding iteration i, the channel detector decodes Y_1^n using the *a priori* information from the LDPC code. In the second half of decoding iteration i, one LDPC decoding iteration is completed using internal edge messages from the previous LDPC iteration and the output of the channel detector. A random scrambling sequence is added to the codeword before transmission and removed before LDPC decoding; this is very similar to using a random coset of the LDPC code. Fig. 6 shows the GTW graph of the joint iterative decoder.

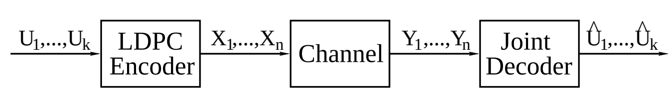

Figure 5. Block diagram of the system.

1) The Dicode Erasure Channel: The dicode erasure channel (DEC) is a binary input channel based on the $1-D$ linear intersymbol-interference (ISI) dicode channel. The output of the $1-D$ channel with binary inputs (e.g., +1, 0, -1) is erased with probability ϵ and transmitted perfectly with probability $1-\epsilon$. More information about the DEC can be found in [16], [11].

The simplicity of the DEC allows the BCJR algorithm for the channel to be analyzed in closed form. The method is similar to the exact analysis of turbo codes on the BEC [2]. The EXIT function for the DEC is computed in [11] and if the outputs are erased with probability ϵ, then the EXIT function of the channel detector is given by

$$f(x;\epsilon)=\frac{4\epsilon^2}{\left(2-x(1-\epsilon)\right)^2}.$$

The capacity of the DEC for independent equiprobable inputs can be computed by analyzing only the forward recursion of the BCJR algorithm [16] and is given by $C_{i.u.d.}(\epsilon)=1-\frac{2\epsilon^2}{1+\epsilon}$.

B. Density Evolution for Joint Decoding

The closed form analysis of this system is based on the fact that all the messages passed in decoding graph are erasure messages. This allows DE of the joint iterative decoder to be represented by a single parameter recursion. Let $f(x;\epsilon)$ be a function which maps the erasure probability, x, of the *a priori* LLR distribution to the erasure probability at the output of the channel detector for a channel erasure probability of ϵ. Following [11], we refer to $f(x;\epsilon)$ as the *extrinsic information transfer* (EXIT) function of the channel.

The joint decoding graph is shown in Fig. 6 and message-passing schedule and variables are shown on the left. Let $x_0^{(l)}$, $x_1^{(l)}$, $x_2^{(l)}$, and $x_3^{(l)}$ denote the erasure rate of messages passed during iteration l. The update equations are as follows

$$\begin{aligned}
x_0^{(l+1)} &= x_3^{(l)}\lambda\left(x_1^{(l)}\right)\\
x_1^{(l+1)} &= 1-\rho\left(1-x_0^{(l+1)}\right)\\
x_2^{(l+1)} &= L\left(x_1^{(l+1)}\right)\\
x_3^{(l+1)} &= f\left(x_2^{(l+1)};\epsilon\right).
\end{aligned}$$

The first two equations simply describe LDPC decoding when the channel erasure parameter is $x_3^{(l)}$ instead of the fixed constant ϵ. The third equation describe the message passing from the code to the channel detector. The fourth equation takes the channel detector bits into account and simply maps side information from the code through the EXIT function $f(x;\epsilon)$.

C. The EXIT Area Theorem for Joint Decoding

In this section, we consider the MAP EXIT function of the entire joint decoder. Consider any FS channel with deterministic ISI that is observed through an erasure channel. In this case, the output sequence Y_1^n consists of independently erased observations (with probability ϵ) of a deterministic sequence Z_1^n that, given the initial state S_1, is in one-to-one correspondence with the input sequence X_1^n.

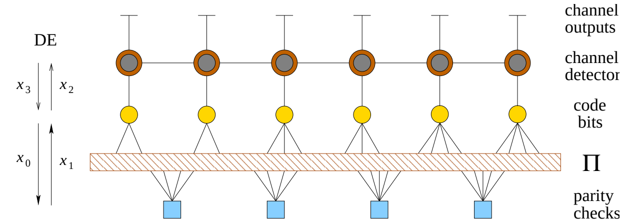

Figure 6. Gallager-Tanner-Wiberg graph of the joint iterative decoder.

Definition 3: Then, the *joint decoding MAP EXIT function* is defined to be

$$h^{MAP-JD}(\epsilon)\triangleq\frac{1}{n}\sum_{i=1}^{n}H\left(Z_i|Y_1^n(\epsilon)\setminus Y_i(\epsilon),S_1\right).$$

Definition 4: Then, the *joint decoding BP EXIT function* is defined to be

$$h^{BP-JD}(\epsilon)\triangleq\frac{1}{n}\sum_{i=1}^{n}h_i^{BP-JD}(\epsilon),$$

where $h_i^{BP-JD}(\epsilon)$ is the entropy of the iterative decoding estimate of Z_i from $Y_1^n\backslash Y_i$ and S_1. The iterative decoding estimate of this output symbol is given by the symbol's extrinsic message in the joint decoder after l iterations of decoding.

Corollary 1: Let X_1^n be chosen according to $p_{X_1^n}(x_1^n)$ and Y_1^n be the result of transmitting X_1^n over the above FS ISI channel. To emphasize that Y_1^n depends on the channel parameter δ write $Y_1^n(\delta)$. Then

$$\frac{1}{n}H\left(X_1^n|Y_1^n(\delta),S_1\right)=\int_0^\delta h^{MAP-JD}(\epsilon)\,d\epsilon.$$

Proof: The proof is a slight modification of the approach taken in [1], [2]. The one-to-one correspondence between X_1^n and Z_1^n (given S_1) implies that

$$\begin{aligned}
H\left(X_1^n|Y_1^n,S_1\right) &= H\left(Z_1^n|Y_1^n,S_1\right)\\
&= H\left(Z_i|Y_1^n,S_1\right)+H\left(Z_1^n\backslash Z_i|Y_1^n,Z_i,S_1\right).
\end{aligned}$$

Since Y_i is a noisy observation of Z_i, we find that $Z_1^n\backslash Z_i\rightarrow Z_i\rightarrow Y_i$ forms a Markov chain. Moreover, if each channel mapping $Z_i\rightarrow Y_i$ depends on a different parameter ϵ_i, then we can write

$$\frac{d}{d\epsilon_i}H\left(X_1^n|Y_1^n,S_1\right)=\frac{d}{d\epsilon_i}H\left(Z_i|Y_1^n,S_1\right)$$

because $H\left(Z_1^n\backslash Z_i|Y_1^n,Z_i,S_1\right)$ is independent of ϵ_i. If we assume also that Y_i is either an erasure (with probability ϵ_i) or $Y_i=Z_i$ (with probability $1-\epsilon_i$), then we can write

$$\begin{aligned}
H\left(Z_i|Y_1^n,S_1\right) &= \epsilon_i H\left(Z_i|Y_1^n\setminus Y_i,S_1\right)\\
&\quad+(1-\epsilon_i)\,H\left(Z_i|Y_1^n\setminus Y_i,Z_i,S_1\right)
\end{aligned}$$

Using the derivative method, this gives

$$\begin{aligned}
\frac{d}{d\epsilon}H\left(X_1^n|Y_1^n,S_1\right) &= \sum_{i=1}^{n}\frac{d}{d\epsilon_i}H\left(Z_i|Y_1^n,S_1\right)\\
&= \sum_{i=1}^{n}H\left(Z_i|Y_1^n\setminus Y_i,S_1\right).
\end{aligned}$$

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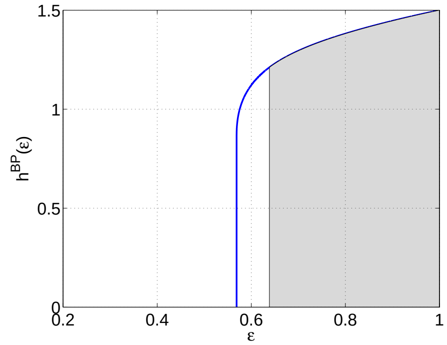

Figure 7. The BP EXIT function $h^{BP}(\epsilon)$ of the joint iterative decoder for a (3,6)-regular LDPC code and the DEC channel. The left boundary of the shaded area is the upper bound on ϵ^{MAP}, and the area of the shaded portion under the curve equals the code rate of $\frac{1}{2}$. This gives $\epsilon^{BP} = 0.5689$ and $\epsilon^{MAP} \le 0.6386$.

D. BP EXIT Function and Bounds on the MAP Threshold

The joint decoding BP EXIT function simply can be computed by analyzing the BCJR decoding algorithm for the DEC. Since the channel has only two states and the channel inputs satisfy $\Pr(X_i = 0) = \Pr(X_i = 1) = \frac{1}{2}$, the forward and backward recursion vectors (which can have infinite support) effectively take only two values; the state is either known (denoted $\mathcal{K}$) or unknown (denoted $\mathcal{U}$) [11].

Lemma 3: The asymptotic BP EXIT function of the joint decoder can be written as

$$\begin{aligned} h^{BP-JD}(\epsilon) &= \Pr(\alpha \in \mathcal{K}, \beta \in \mathcal{U}) \cdot \delta + \Pr(\alpha \in \mathcal{U}, \beta \in \mathcal{K}) \\ &\quad + \Pr(\alpha \in \mathcal{U}, \beta \in \mathcal{U}) \cdot \left[\frac{3}{2}\delta + (1-\delta)\right] \\ &= \frac{2\epsilon\delta\,(4 + \epsilon\delta - 2\delta)}{(2 - \delta(1-\epsilon))^2}, \end{aligned}$$

where $\delta = L(x_1)$ and x_1 is given by the $l \to \infty$ DE fixed point for ϵ.

Proof: To compute the BP EXIT function, we can simply analyze the output stage of the BCJR algorithm and compute the entropy of Z_i given the current messages. Notice that, at any point in the trellis, there are four distinct possibilities for forward/backward recursion (α/β) state knowledge: $(\mathcal{K}/\mathcal{K})$, $(\mathcal{K}/\mathcal{U})$, $(\mathcal{U}/\mathcal{K})$, and $(\mathcal{U}/\mathcal{U})$. Let ϵ be the channel erasure rate and $\delta = L(x_1)$ be the *a priori* erasure rate from the LDPC code, and define $H_{AB} = H(Z_i|\alpha_i \in A, \beta_{i+1} \in B)$. Examining the trellis shows that

$$\begin{aligned} H_{\mathcal{K}\mathcal{K}} &= 0 & H_{\mathcal{K}\mathcal{U}} &= \delta \\ H_{\mathcal{U}\mathcal{K}} &= 1 & H_{\mathcal{U}\mathcal{U}} &= \frac{3}{2}\delta + (1-\delta). \end{aligned}$$

From [11], the steady state probability that the forward/backward recursion has no state knowledge is

$$\begin{aligned} \Pr(\alpha \in \mathcal{U}) &= \frac{2\epsilon\delta}{2 - \delta(1+\epsilon) + 2\epsilon\delta} \\ \Pr(\beta \in \mathcal{U}) &= \frac{2\epsilon}{(1-\epsilon)(2-\delta) + 2\epsilon}. \end{aligned}$$

■

Fig. 7 shows the BP EXIT function $h^{BP}(\epsilon)$, and area under the BP EXIT function curve equals the code rate $\frac{1}{2}$. The left boundary of the integration area is the upper bound on ϵ^{MAP} of the joint iterative decoder for a (3,6)-regular LDPC code and the DEC channel.

V. Conclusions

In this paper, upper bounds on the MAP thresholds are computed for three iterative decoding systems: IRA codes, ARA codes, and the joint decoding of LDPC codes and channels with memory. These bounds are based on the techniques introduced by Méasson, Montanari, and Urbanke in [1], [2]. The bound for joint decoding requires a slight generalization of the EXIT area theorem that is introduced within.

Some open questions include the tightness of these bounds and the existence of simpler (or closed-form) expressions. These bounds also have natural extensions to non-erasure channel by way of the generalized EXIT (GEXIT) functions introduced in [17].

References

[1] C. Méasson, A. Montanari, and R. Urbanke, "Maxwell's construction: The hidden bridge between iterative and maximum a posteriori decoding," *Arxiv preprint cs.IT/0506083v1*, June 2005.

[2] C. Méasson, "Conservation laws for coding," Ph.D. dissertation, Swiss Federal Institute of Technology, Lausanne, 2006.

[3] S. ten Brink, "Convergence of iterative decoding," *Electronic Letters*, vol. 35, no. 10, pp. 806–808, May 1999.

[4] T. J. Richardson, M. A. Shokrollahi, and R. L. Urbanke, "Design of capacity-approaching irregular low-density parity-check codes," *IEEE Trans. Inform. Theory*, vol. 47, no. 2, pp. 619–637, Feb. 2001.

[5] A. Ashikhmin, G. Kramer, and S. ten Brink, "Extrinsic information transfer functions: model and erasure channel properties," *IEEE Trans. Inform. Theory*, vol. 50, no. 11, pp. 2657–2674, Nov. 2004.

[6] H. Jin, A. Khandekar, and R. J. McEliece, "Irregular repeat-accumulate codes," in *Proc. Int. Symp. on Turbo Codes & Related Topics*, Brest, France, Sept. 2000.

[7] A. Abbasfar, D. Divsalar, and Y. Kung, "Accumulate repeat accumulate codes," in *Proc. IEEE Int. Symp. Information Theory*, Chicago, IL, June 2004, p. 505.

[8] H. D. Pfister and I. Sason, "Accumulate–repeat–accumulate codes: Capacity-achieving ensembles of systematic codes for the erasure channel with bounded complexity," *IEEE Trans. Inform. Theory*, vol. 53, no. 6, pp. 2088–2115, June 2007.

[9] C. Douillard, M. Jézéquel, C. Berrou, A. Picart, P. Didier, and A. Glavieux, "Iterative correction of intersymbol interference: Turbo equalization," *Eur. Trans. Telecom.*, vol. 6, no. 5, pp. 507–511, Sept. – Oct. 1995.

[10] B. M. Kurkoski, P. H. Siegel, and J. K. Wolf, "Joint message-passing decoding of LDPC codes and partial-response channels," *IEEE Trans. Inform. Theory*, vol. 48, no. 6, pp. 1410–1422, June 2002.

[11] H. D. Pfister and P. H. Siegel, "Joint iterative decoding of LDPC codes for channels with memory and erasure noise," *IEEE J. Select. Areas Commun.*, vol. 26, no. 2, pp. 320–337, Feb. 2008.

[12] C. Méasson, A. Montanari, and R. Urbanke, "Maxwell's construction: the hidden bridge between maximum-likelihood and iterative decoding," in *Proc. IEEE Int. Symp. Information Theory*, Chicago, IL, June 2004, p. 225.

[13] M. G. Luby, M. Mitzenmacher, M. A. Shokrollahi, and D. A. Spielman, "Efficient erasure correcting codes," *IEEE Trans. Inform. Theory*, vol. 47, no. 2, pp. 569–584, Feb. 2001.

[14] T. J. Richardson and R. L. Urbanke, "The capacity of low-density parity-check codes under message-passing decoding," *IEEE Trans. Inform. Theory*, vol. 47, no. 2, pp. 599–618, Feb. 2001.

[15] A. Montanari, "Tight bounds for LDPC and LDGM codes under MAP decoding," *IEEE Trans. Inform. Theory*, vol. 51, no. 9, pp. 3221–3246, 2005.

[16] H. D. Pfister, "On the capacity of finite state channels and the analysis of convolutional accumulate-m codes," Ph.D. dissertation, University of California, San Diego, La Jolla, CA, USA, March 2003.

[17] C. Measson, A. Montanari, T. Richardson, and R. Urbanke, "The generalized area theorem and some of its consequences," *Arxiv preprint cs.IT/0511039*, 2005.