Capacity-Achieving Ensembles for the Binary Erasure Channel With Bounded Complexity

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Abstract — **We present two sequences of ensembles of non-systematic irregular repeat-accumulate codes which asymptotically (as their block length tends to infinity) achieve capacity on the binary erasure channel (BEC) *with bounded complexity.* This is in contrast to all previous constructions of capacity-achieving sequences of ensembles whose complexity grows at least like the log of the inverse of the gap to capacity. The new bounded complexity result is achieved by allowing a sufficient number of state nodes in the Tanner graph representing the codes.**

I. Introduction and Outline of the Work

During the last decade, there have been exciting developments in the construction of low-complexity error-correction codes which closely approach the capacity of many standard communication channels with feasible complexity. The first capacity-achieving degree distributions for the binary erasure channel (BEC) were found by Luby et al. [3].

In [2], Khandekar and McEliece discussed the complexity of achieving the channel capacity on the BEC, and more general channels. They conjectured that if the achievable rate under message-passing iterative (MPI) decoding is a fraction $1-\varepsilon$ of the channel capacity, then for a wide class of channels, the encoding complexity scales like $\ln \frac{1}{\varepsilon}$ and the decoding complexity scales like $\frac{1}{\varepsilon} \ln \frac{1}{\varepsilon}$. This conjecture is based on the assumption that the number of edges (per information bit) in the associated bipartite graph scales like $\ln \frac{1}{\varepsilon}$, and the required number of iterations under MPI decoding scales like $\frac{1}{\varepsilon}$. However, for codes defined on graphs which are transmitted over a BEC, the decoding complexity under the MPI algorithm behaves like $\ln \frac{1}{\varepsilon}$ (same as encoding complexity) [3, 4, 5]. This is since the absolute reliability provided by the BEC allows every edge in the graph to be used only once during MPI decoding.

In [4], Sason and Urbanke considered the question of how sparse can parity-check matrices of binary linear codes be, as a function of their gap (in rate) to capacity (where this gap depends on the channel and the decoding algorithm). If the code is represented by a standard Tanner graph without state nodes, the decoding complexity under MPI decoding is strongly linked to the density of the corresponding parity-check matrix (i.e., the number of edges in the graph per information bit). In particular, they consider an arbitrary sequence of binary linear codes which achieves a fraction $1-\varepsilon$ of the capacity of a binary-input output-symmetric memoryless channel with vanishing bit error probability. By information-theoretic tools, they proved [4, Theorem 2.1] that for every such sequence of codes and for every sequence of parity-check matrices which represent these codes, the asymptotic density of these parity-check matrices grows at least like $\frac{K_1+K_2 \ln \frac{1}{\varepsilon}}{1-\varepsilon}$ where K_1 and K_2 only depend on the channel. It is important to mention that this bound is valid under ML decoding, and hence, also for every sub-optimal decoding algorithm. The tightness of the lower bound for the BEC was demonstrated in [4, Theorem 2.3] by analyzing the capacity-achieving sequence of right-regular LDPC-code ensembles introduced by Shokrollahi [5]; the analysis assumes MPI decoding. Based on the discussion in [4], it follows that for every iterative decoder which is based on the standard Tanner graph, there exists a fundamental tradeoff between performance and complexity, and the complexity (per information bit) becomes *unbounded* when the gap between the achievable rate and the channel capacity vanishes. Therefore, it was suggested in [4] to study if better tradeoffs can be achieved by allowing more complicated graphical models (e.g., graphs which also involve state nodes).

In this paper, we present sequences of capacity-achieving ensembles for the BEC with bounded complexity under MPI decoding. The new ensembles are non-systematic irregular repeat-accumulate (IRA) codes with properly chosen degree distributions (for background on ensembles of IRA codes, the reader is referred to [1]). The new bounded complexity results improve on the results in [6], and demonstrate the superiority of properly designed non-systematic IRA codes over systematic IRA codes (since the complexity of any sequence of ensembles of systematic IRA codes becomes *unbounded* under MPI decoding when the gap between the achievable rate and the capacity vanishes [6]). As predicted by [4], the new bounded complexity result is achieved by allowing a sufficient number of state nodes in the Tanner graph representing the codes. In particular, we suggest a sequence of capacity-achieving ensembles of non-systematic IRA codes where the degree of the check nodes is 5, so the complexity per information bit under MPI decoding is upper bounded by $\frac{5}{1-p}$ (p designates the bit erasure probability of the BEC). Computer simulation results for these ensembles appear to agree with this analytical result.

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