Reed–Muller Codes Achieve Capacity on BMS Channels

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Abstract

This paper considers the performance of long Reed–Muller (RM) codes transmitted over binary memoryless symmetric (BMS) channels under bitwise maximum-a-posteriori decoding. Its main result is that the family of binary RM codes achieves capacity on any BMS channel with respect to bit-error rate. This resolves a long-standing open problem that connects information theory and error-correcting codes. In contrast with the earlier result for the binary erasure channel, the new proof does not rely on hypercontractivity. Instead, it combines a nesting property of RM codes with new information inequalities relating the generalized extrinsic information transfer function and the extrinsic minimum mean-squared error.

Keywords. BMS channels, capacity-achieving codes, GEXIT functions, linear codes, MAP decoding, MMSE, Reed-Muller codes.

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1 Introduction

Reed–Muller (RM) codes have been the subject of considerable research since their introduction by Muller in [1] and their majority-logic decoding by Reed in [2]. Almost 70 years after their discovery, RM codes remain an active area of research in theoretical computer science and coding theory. In 2007, Costello and Forney described “the road to channel capacity” [3] and wrote that:

[I]n recent years it has been recognized that “RM codes are not so bad”. RM codes are particularly good in terms of performance versus complexity with trellis-based decoding and other soft-decision decoding algorithms [...]

Indeed, with optimum decoding, RM codes may be “good enough” to reach the Shannon limit on the AWGN channel. [...] It seems likely that the real coding gains of the self-dual RM codes with optimum decoding approach the Shannon limit [...], but to our knowledge this has never been proved.

We note that their observations preceded the introduction of polar codes [4] by roughly one year and, after polar codes, there was a significant renaissance in research on RM codes [5, 6]. This paper considers the performance of long RM codes transmitted over binary memoryless symmetric (BMS) channels under bitwise maximum-a-posteriori (MAP) decoding and proves that their intuition was indeed correct.

For a BMS channel, the output sequence is generated by passing all symbols in the input sequence through independent identically-distributed channels whose noise processes do not depend on the input symbol. Some examples are the binary erasure channel (BEC), the binary symmetric channel (BSC), and the binary-input additive white Gaussian noise (BIAWGN) channel. The primary technical result can be summarized by the following theorem.

**Theorem 1.** Consider any BMS channel with capacity \( C \in (0, 1) \). For every sequence of RM codes with strictly increasing blocklength and rate converging to \( R \in [0, C) \), the bit-error rate (BER) under bit-MAP decoding converges to zero.

This settles a rather old question in coding theory and shows that binary Reed–Muller codes can achieve capacity on any BMS channel under bit-MAP decoding! We note that this conclusion was certainly more expected than its alternative because [7] already established this result for the special case of the BEC. For a detailed discussion of relevant prior work until 2017, see [7]. Since then, there have been a few papers that address this question directly or indirectly [8, 9, 10]. In particular, the results of [10] use the BEC result from [7] to establish that RM codes can decode successfully on general
BMS channels but only at rates bounded away from capacity. One can also find a good tutorial overview of RM codes and recent results in [11].

The proof for the BEC case in [7] requires only linearity and doubly-transitive symmetry for the code. To achieve this, it relies on the sharp threshold property for symmetric boolean functions and the Extrinsic Information Transfer (EXIT) Area Theorem [7]. One new element in this work is that our proof also relies on the RM nesting property which says that longer RM codes can be punctured down to shorter RM codes of the same order. But, this does not follow directly from the doubly-transitive symmetry of the code. Another difference between the new proof and [7] is highlighted by the fact that the new proof holds for the BEC but does not make use of hypercontractivity (which seems to be crucial for the boolean function result). Lastly, the new proof does not extend to all sequences of doubly-transitive codes nor does it imply that the block-error rate converges to 0. However, we are optimistic that an extension to block-error rate is possible, perhaps using techniques from [12, 13, 10].

There has also been significant recent interest in finding low-complexity decoders for RM codes with near-optimal performance [14, 15, 16, 17, 18, 19, 20, 21, 22]. We do not delve into this work but mention only that many of these approaches also exploit the symmetry and nesting properties of RM codes.

1.1 Primary Contributions and Overview

The main contribution of this paper is to establish that RM codes can achieve capacity for any BMS channel. Our proof uses many ideas developed previously in the context of generalized EXIT (GEXIT) analysis. For example, we focus on a family of BMS channels and use the GEXIT area theorem. However, there are a number of steps in our proof that appear to be new. Since these steps may be of interest in their own right we summarize them briefly here.

A major theme in this work is our focus on the impact of extra observations on the estimate of a single codeword bit (e.g., see Lemma 24). The precise form of the “extra observation” varies from place to place. In some cases it corresponds to a second look at a single position in the codeword and in other cases it corresponds to a second look at a collection of symbol positions. But the underlying idea is the same — an additional observation cannot make a meaningful difference in the ability to estimate the bit of interest if either of the following conditions is met:

(i) the expected information from the first observation is very small so that a second independent observation is unlikely to tell us much more; or

(ii) the expected information from the first observation is nearly maximal and a second independent observation cannot contribute much more.

To fully utilize this observation, a crucial step is harnessing the nesting property of RM codes to provide a strong upper bound on the variance of the conditional mean of a codeword bit given the observation (e.g., see Section 5.2.2). In particular, we embed the RM code of interest in a longer RM code with a slightly lower rate and show that the two codes must behave very similarly for almost all channel noise parameters as the block length grows.

To make these arguments precise, one needs to compare the associated GEXIT functions with and without extra observations. While there are numerous functional properties associated with mutual information and entropy in the context of an additional observation, the challenge faced in our setting is that the GEXIT function corresponds to a difference in mutual information, and in this setting many of the usual properties no longer hold.

The technical tools that allow us to overcome this difficulty form a collection of generalized I-MMSE relations, which are introduced in Section 4.2. They allow us to bound the GEXIT function in terms of a quantity, called the extrinsic MMSE, which is easier to analyze. In particular, the extrinsic MMSE satisfies a data processing inequality and has a sub-additivity property, which follows as a natural consequence of the Efron-Stein inequality.

Here is a list of key elements in the proof along with brief descriptions:

- Lemma 8 describes the RM nesting property as used in the proof.
- Lemma 17 derives the two-look formula, which is the foundation for our GEXIT analysis.
• Lemmas 20 and 21 use the two-look formula to relate the GEXIT function to the extrinsic MMSE.
• Lemma 27 derives an integral constraint on the extrinsic MMSE function of RM codes that shows it must transition quickly from 0 to 1 as the blocklength increases.
• Lemma 35 uses the GEXIT area theorem to compare the transition point of the extrinsic MMSE to the capacity of the BMS channel.
• Theorem 36 proves the main result by deriving a non-asymptotic upper bound on BER of an RM code on any BMS channel.

For readers who are familiar with [7] and boolean functions, Appendix A.3 discusses the method introduced in this paper, for the special case of the BEC, and compares it with the approach in [7].

1.2 Other Consequences
Our work also has some additional consequences when combined with other recent results:
• Since our main result shows that RM codes achieve capacity on the BSC, it follows that Quantum Reed–Muller (QRM) codes [23] achieve the hashing bound on a modified depolarizing channel where $X$ and $Z$ errors occur independently with the same probability [24, p. 568].
• Combined with the duality result of Renes for classical-quantum (CQ) channels [25], our result shows that an RM code on a pure-state CQ channel can allow reliable detection of a few code symbols up to the channel capacity limit. Due to the sequential nature of the implied quantum detection model, however, the decay rate of error probability that we establish is not sufficient to guarantee a vanishing bit-error rate for all code symbols. If the upper bound on the bit-error probability can be improved to decay quickly enough, then the quantum union bound can show that block-error probability vanishes for the pure-state CQ channel. In that case, Renes’s duality implies additionally that RM codes can achieve strong secrecy for both the BSC and the pure-state wire-tap channel under optimal decoding.

1.3 Notation
The real numbers and extended real numbers are denoted by $\mathbb{R}$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\}$. The natural numbers are denoted by $\mathbb{N} := \{1, 2, \ldots \}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $N \in \mathbb{N}_0$, a range of natural numbers is denoted by $[N] := \{0, 1, \ldots, N-1\}$. Also, $\mathbb{F}_2$ is used to denote the Galois field with 2 elements (i.e., the integers $\{0, 1\}$ with addition and multiplication modulo 2). For a set $\mathcal{X}$, the vector $x \in \mathcal{X}^N$ has length $N$ and is indexed from 0 so that $x = (x_0, \ldots, x_{N-1})$. For an $M$-element index set $A = \{a_0, a_1, \ldots, a_{M-1}\} \subseteq [N]$ with $a_0 < a_1 < \cdots < a_{M-1}$, we define the subvector $x_A = (x_{a_0}, x_{a_1}, \ldots, x_{a_{M-1}}) \in \mathcal{X}^M$.

2 Reed–Muller Codes
2.1 Background
A length-$N$ binary code is a set $\mathcal{C} \subseteq \{0, 1\}^N$ of length-$N$ binary vectors called codewords. Such a code allows the transmission of $|\mathcal{C}|$ different messages each of which is associated with a codeword $c \in \mathcal{C}$. A codeword $c = (c_0, \ldots, c_{N-1})$ is transmitted using a sequence of channel uses where the $i$-th code symbol, $c_i$, determines the input for the $i$-th channel use. The rate of the code $\mathcal{C}$ is defined to be $\frac{1}{N} \log_2 |\mathcal{C}|$.

For a length-$N$ binary linear code $\mathcal{C}$ with dimension $K$, the code rate equals $K/N$ and a generator matrix $G \in \mathbb{F}_2^{K \times N}$ defines an encoder $E : \mathbb{F}_2^K \rightarrow \mathcal{C}$ that maps an information vector $u \in \mathbb{F}_2^K$ to a codeword via $u \rightarrow uG$. The Reed–Muller code $\text{RM}(r, m)$ is a binary linear code of length $N = 2^m$ and rate

$$R(r, m) := \frac{1}{2^m} \sum_{i=0}^{r} \binom{m}{i}. \tag{1}$$

Below, we introduce facts about RM codes as they are needed. For a thorough discussion, see [26, 27].
Example 2. Let $G_{r,m}$ be the generator matrix of $\text{RM}(r,m)$. The generator matrix (in the standard RM order) of $\text{RM}(1,3)$ is given by

$$G_{1,3} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} G_{1,2} \\ G_{0,2} \end{bmatrix},$$

where

$$G_{1,2} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } G_{0,2} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$  

RM codes can be described in many different ways. One way is via the one-to-one correspondence between the set of codewords in $\text{RM}(r,m)$ and the set of $\mathbb{F}_2$-multilinear polynomials in $m$ indeterminates whose total degree is at most $r$. For this correspondence, the mapping from a polynomial $p$ to a codeword $c$ is given by evaluating $p$ at all points $v \in \mathbb{F}_2^m$. In particular, the $i$-th code symbol is given by $c_i = p(v)$ where $v = (v_0, \ldots, v_{m-1}) \in \mathbb{F}_2^m$ is the binary expansion of $i = \tau(v)$ and

$$\tau(v) := \sum_{j \in [m]} v_j 2^j.$$  

For $0 \leq r \leq m$, let $\mathcal{P}_{r,m}$ be the vector space (over $\mathbb{F}_2$) of multilinear polynomials in $m$ indeterminates with degree at most $r$. This vector space is spanned by the subset of multilinear monomials

$$\mathcal{M}_{r,m} := \left\{ \prod_{j \in S} v_j \big| S \in \mathcal{S}_{r,m} \right\},$$

where $\mathcal{S}_{r,m} := \{S \subseteq [m] \mid |S| \leq r\}$. Each polynomial in $\mathcal{P}_{r,m}$ is defined by a set of coefficients $\{\alpha_S \in \mathbb{F}_2\}_{S \in \mathcal{S}_{r,m}}$ with respect to the monomial basis and its evaluation is given by

$$v \mapsto \sum_{S \in \mathcal{S}_{r,m}} \alpha_S \prod_{j \in S} v_j.$$  \hfill (2)

This viewpoint can be unified with the generator matrix perspective by noting that, for $S \in \mathcal{S}_{r,m}$, the coefficient $\alpha_S$ can be seen as an information bit that modulates the row in the generator matrix associated with the monomial $S$. In particular, that row can be computed by evaluating the monomial $S$ at all points in $\mathbb{F}_2^m$. To generate the codebook, one first enumerates all information vectors $u \in \mathbb{F}_2^K$ (or equivalently all polynomials in $\mathcal{P}_{r,m}$) and then multiplies each by $G$ (or equivalently each polynomial is evaluated at the $2^m$ points in $\mathbb{F}_2^m$).

Example 3. Continuing Example 2, we observe that the only degree-0 monomial is 1. Thus, the first and only row of $G_{0,2}$ can be computed by evaluating $p(v) = 1$ for all $v \in \{0,1\}^2$. This also explains the first row of $G_{1,2}$. The second and third rows of $G_{1,2}$ are associated with evaluating $p(v) = v_0$ and $p(v) = v_1$ for all $v \in \{0,1\}^2$ in the order given by $\tau(v)$. Likewise, for $G_{1,3}$, the rows are associated with evaluating the monomials $\{1, v_0, v_1, v_2\}$, respectively, for all $v \in \{0,1\}^3$ with the order given by $\tau(v)$.

RM codes have many algebraic and combinatorial properties. One of these is a nesting property that will play a particularly important role in this work. To describe this property, we will consider different ways that $\mathcal{C} = \text{RM}(r,m)$ can be punctured down to the code $\text{RM}(r,m-k)$.

Definition 4 (Punctured Code). For a length-$N$ binary code $C$ and a subset $I \subseteq [N]$, we denote by $C_I$ the punctured code formed by only keeping symbol indices with positions in $I$. Formally, we write

$$C_I := \{c' \in \{0,1\}^{|I|} : \exists c \in C, c_I = c'\}.$$
The code $C = \text{RM}(r, m)$ can be punctured down to $\text{RM}(r, m - k)$ in two different ways. In other words, there exist $I, I' \subseteq \{0,1\}^m$ such that $C_I$ and $C_{I'}$ are both equal to $\text{RM}(r, m - k)$. We will see below that this statement follows naturally from two well-known properties of RM codes. The first property is encapsulated in the following lemma.

**Lemma 5 (RM Puncturing).** If one punctures the code $C = \text{RM}(r, m)$ by only keeping symbol positions with indices in the set $I = \tau(V)$, where $V = \{0,1\}^{m-k} \times \{0\}^{k}$, then $C_I = \text{RM}(r, m - k)$. Moreover, puncturing a uniform random codeword from $C$ results in a uniform random codeword from $C_I$.

**Proof.** To see this, we can split the monomials into two groups. Let the first set of monomials be the

$(1\to m)$

subset of $\mathcal{M}_{r,m}$ that only contains the variables $v_0, \ldots, v_{m-k}$ and observe that this equals $\mathcal{M}_{r,m-k}$. That means the second set, which contains all the rest, is given by $\mathcal{M}' = \mathcal{M}_{r,m} \setminus \mathcal{M}_{r,m-k}$. The key observation is that the monomials in $\mathcal{M}'$ all evaluate to 0 on the set $V$ because all points in $V$ have $v_{m-k} = \ldots = v_{m-1} = 0$. Thus, for $c \in C$ and $i \in I$, only the monomials in $\mathcal{M}_{r,m-k}$ contribute to the value of $c_i$. This implies that the codewords in $C_I$ are formed by evaluating the set of $\mathbb{F}_2$-multilinear polynomials in $m - k$ indeterminates whose total degree is at most $r$ at the points in $V$. Hence, $C_I$ is precisely equal to $\text{RM}(r, m - k)$. Another important point is that exactly $2^{\lvert \mathcal{M}' \rvert} = |C|/|C_I|$ codewords in $C$ are mapped to each codeword in $C_I$. This holds because, if the information bits associated with $\mathcal{M}_{r,m-k}$ are fixed, then the punctured codeword $c_I$ is fixed. But, by choosing the information bits associated with the monomials in $\mathcal{M}'$, one can generate $2^{\lvert \mathcal{M}' \rvert}$ different codewords in $C$ that have the same $c_I$. $\square$

The second property is that, for an invertible binary matrix $Q \in \mathbb{F}_2^{m \times m}$ and a vector $b \in \mathbb{F}_2^m$, the degree of a polynomial is preserved by the affine change of variables $v \mapsto \pi_{Q,b}(v)$ where $\pi_{Q,b} : \{0,1\}^m \to \{0,1\}^m$ is defined by

$$\pi_{Q,b}(v) = \sum_{j=1}^m Q_{i,j}v_j + b_j.$$  

Thus, the set of all multilinear polynomials with degree at most $r$ is mapped to itself by this change of variables and the permutation $\tau(\pi_{Q,b}(\tau^{-1}(i)))$ defines an automorphism of the RM code in terms of symbol indices [26, p. 398].

Combining these two properties, one finds that, if an evaluation subset $V \subseteq \{0,1\}^m$ is an $\mathbb{F}_2$-subspace with dimension $m - k$, then there is an invertible binary matrix $Q$ (and hence a linear automorphism $\pi_{Q,0}$) that maps $V$ to $\{0,1\}^{m-k} \times \{0\}^k$. Thus, each $m - k$ dimensional subspace $V \subseteq \{0,1\}^m$ defines a puncturing pattern $I = \tau(V)$ that reveals an $\text{RM}(r, m - k)$ code inside an $\text{RM}(r, m)$ code. Moreover, all of these punctured codes contain the code symbol $c_0$ because all of these automorphisms map 0 to 0.

**Example 6.** Continuing the example, we observe that the generator matrix decomposition in Example 2 implies that, for all $c \in \text{RM}(1, 3)$, we have $(c_0, c_1, c_2, c_3) \in \text{RM}(1, 2)$. This also follows from evaluating all degree at most 1 polynomials in 3 indeterminates on the set $V = \text{span}\{(1,0,0),(0,1,0)\}$. This gives the first 4 symbols of all codewords in $\text{RM}(1, 3)$ because $I = \tau(V) = \{0,1,2,3\}$. We can also extract columns 0, 1, 4, 5 from $G_{1,3}$ to get the submatrix

$$G' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$ 

![Diagram showing two copies of RM(1, 2) inside RM(1, 3) based on Example 6](image-url)
Since the row space of $G'$ equals the row space of $G_{1,2}$, we observe that any codeword of $\text{RM}(1,3)$ also contains a codeword of $\text{RM}(1,2)$ in these bit positions. Alternatively, we could evaluate the above set of polynomials on the set $V' = \text{span}\{(1,0,0),(0,0,1)\}$. This calculation gives the code symbols indexed by $0,1,4,5$ for all codewords of $\text{RM}(1,3)$ because $I' = \pi(V') = \{0,1,4,5\}$. Due to symmetry, however, we will obtain the same set of codewords as we obtained by evaluating on the set $V$. Thus, we see again that for all $c \in \text{RM}(1,3)$ we have $(c_0,c_1,c_4,c_5) \in \text{RM}(1,2)$. Figure 7 illustrates this example using the notation defined in Lemma 8 (which outlines a general version of this construction).

For an $\text{RM}(r,m)$ code, the number of information symbols is equal to

$$K = |S_{r,m}| = \sum_{i=0}^{r} \binom{m}{r}$$

because (2) implies that we can assign one information bit to each $\alpha_S$ for $S \in S_{r,m}$. This information symbol determines whether or not the monomial defined by $S$ is present and the associated polynomial. This justifies the rate formula in (1). Notice that the rate formula equals the cdf of a binomial random variable, with $m$ equiprobable trials, evaluated at $r$. For large $m$, the central limit theorem implies that $R(r,m)$ transitions from roughly 0.025 to roughly 0.975 as $r$ ranges from $\lfloor m/2 - \sqrt{m} \rfloor$ to $\lfloor m/2 + \sqrt{m} \rfloor$ because the standard deviation of the binomial is $\sqrt{m}/2$ and, for a Gaussian, roughly 95% percent of the probability lies within 2 standard deviations of the mean. It can also be useful to consider sequences of RM codes where the $n$-th code is $\text{RM}(r_n,m_n)$ with $m_n \to \infty$ and $r_n = m_n/2 + \alpha\sqrt{m_n}/2 + O(\sqrt{m_n})$. In particular, for such sequences, the central limit theorem implies that $R(r_n,m_n) \to \Phi(\alpha)$, where \[ \Phi(\alpha) = (2\pi)^{-1/2} \int_{-\infty}^{\alpha} \exp(-z^2/2) \, dz \] is the cumulative distribution function of a standard Gaussian random variable.

The above rate calculation appeared earlier in [7, Remark 24] and we mention it here for completeness. There, it is observed that, for any code rate $R \in (0,1)$, the rate calculation implies one can construct a sequence of RM codes with increasing $m$ whose code rate converges to $R$.

### 2.2 New Observations

The following lemma characterizes the change in code rate due to perturbations of the $m$ parameter.

**Lemma 7 (RM Rate Change).** For the codes $\text{RM}(r,m)$ and $\text{RM}(r,m+k)$ with $k \geq 1$, we have

$$R(r,m) - R(r,m+k) \leq \frac{3k + 4}{5\sqrt{m}}.$$

**Proof.** Recall that $R(r,m)$ is equal to the cdf, evaluated at $r$, of the sum of $m$ independent symmetric Bernoulli random variables. Sharp bounds on the normal approximation for the symmetric binomial distribution show that $|R(r,m) - \Phi(\alpha(r,m))| \leq 1/(2\pi m)$ where $\alpha(r,m) := (2r - m)/\sqrt{m}$ and $\Phi(z) := (2\pi)^{-1/2} \int_{-\infty}^{z} \exp(-u^2/2) \, du$ is the cdf of the standard Gaussian distribution [28, Corollary 1.2]. Using two applications of this bound, we obtain

$$R(r,m) - R(r,m+k) \leq \Phi(\alpha(r,m)) - \Phi(\alpha(r,m+k)) + \frac{1}{\sqrt{2\pi m}} + \frac{1}{\sqrt{2\pi(m+k)}}.$$

To bound the difference between the Gaussian cdfs, we can write

$$\Phi(\alpha(r,m)) - \Phi(\alpha(r,m+k)) = \int_{m+k}^{m} \frac{d\Phi(\alpha(r,x))}{dx} \, dx = \int_{m}^{m+k} \frac{x + 2r}{2x^{3/2}} \Phi'(\alpha(r,x)) \, dx \leq \frac{1}{\sqrt{2\pi}} \int_{m}^{m+k} \frac{x + 2r}{2x^{3/2}} \, dx \leq \frac{1}{\sqrt{2\pi}} \frac{k(m + 2r)}{2m^{3/2}},$$

where we use $\Phi'(z) \leq 1/\sqrt{2\pi}$ in the first inequality and the fact that the integrand is non-increasing in the second inequality. Noting that $r \leq m$, we can simplify to get the bound

$$R(r,m) - R(r,m+k) \leq \frac{3k}{2\sqrt{2\pi m}} + \frac{2}{\sqrt{2\pi m}} = \frac{3k + 4}{\sqrt{8\pi m}} < \frac{3k + 4}{5\sqrt{m}},$$

where the last step follows from $\sqrt{8\pi} > 5$. \qed
The above observations have a surprising consequence that, as far as we know, has not been exploited previously. Notice that, if the code sequence RM(r, m, n) satisfies R(r, m, n) \rightarrow R for R \in (0,1), then the rate of the code sequence RM(r_n, m_n + k_n) also converges to R for k_n = o(\sqrt{m_n}). But, RM(r_n, m_n) can be formed from RM(r_n, m_n + k_n) by puncturing all but the first 2^m_n symbols. Thus, we have a code sequence whose rate converges to R where throwing away a fraction 1 - 2^{-\kappa_n} of the symbols gives another code sequence whose rate converges to R. This is quite surprising because one would expect that puncturing a significant fraction of the bits should increase the code rate by a significant amount.

**Lemma 8.** For an RM(r, m+k) code C with r \leq m and 1 \leq k \leq m, there are multiple distinct puncturing patterns that result in an RM(r, m) code. In particular, there are subsets A, B, C \subset [2^{m+k}] that define puncturing patterns I = \{0\} \cup A \cup B and I' = \{0\} \cup A \cup C such that C_I = C_{I'} = \text{RM}(r, m)\). In addition, |I \cap I'| = 2^{m-k}, C_{I \cap I'} = \text{RM}(r, m-k), and a uniform distribution on RM(r, m + k) induces a uniform distribution on the punctured codes C_I and C_{I'}. This construction is shown in Figure 2.

**Proof.** Let V = \mathbb{F}_2^m \times \{0\}^k (i.e., all length-m + k binary vectors where the last k entries are zero) be the subspace of \mathbb{F}_2^{m+k} spanned by the first m canonical basis vectors and let I = \tau(V) = [2^m] be the associated set of codeword indices. This implies that C_I is given by evaluating the set of degree at most r polynomial in m + k variables on the subset V. Using Lemma 5 we see that C_I is equal to \text{RM}(r, m).

Similarly, let V' = \mathbb{F}_2^{m-k} \times \{0\}^k \times \mathbb{F}_2^k (i.e., all length-m + k binary vectors satisfying v_{m-k} = v_{m-k+1} = \cdots = v_{m-1} = 0) be the subspace of \mathbb{F}_2^{m+k} spanned by the first m - k and last k canonical basis vectors. For the associated set of codeword indices, I' = \tau(V'), this implies that C_{I'} is given by evaluating the set of degree at most r polynomial in m + k variables on the subset V'. By the previous arguments in this section, C_{I'} is equal to \text{RM}(r, m). We note that the condition k \geq 1 is necessary to avoid the k = 0 case where V = V'.

Since V \cap V' = \mathbb{F}_2^{m-k} \times \{0\}^{2^k} is a subspace with 2^{m-k} points, it follows that T := I \cap I' = \tau(V \cap V') = [2^{m-k}] and C_T is equal to \text{RM}(r, m-k). Then, we can define the set A = T \setminus \{0\} to be the non-zero overlap, the set B = I \setminus T to be the indices needed to complete I, and the set C = I' \setminus T to be the indices needed to complete I'. Finally, as noted in Lemma 5 a uniform distribution on C generates a uniform distribution C_I (and hence C_{I'}).

The following example hints at how this property can be exploited to analyze RM codes.

**Example 9.** Continuing Example 6, consider the case where a random codeword X = (X_0, \ldots, X_7) \in \text{RM}(1, 3) is transmitted over a BEC and received as Y = (Y_0, \ldots, Y_7) \in \{0, 1, ?\}^7. Then, one can estimate X_0 from Y_1, Y_2, Y_3 using only the fact that (X_0, X_1, X_2, X_3) \in \text{RM}(1, 2). One can also estimate X_0 from Y_1, Y_4, Y_5 using only the fact that (X_0, X_1, X_4, X_5) \in \text{RM}(1, 2). In this case, X_0 will be recovered if either of these estimates is not an erasure. Moreover, the performance given by combining the two estimates is strictly better than that given by either single estimate unless the two estimates are perfectly correlated. See Figure 2 for a graphical representation of this construction.

Since there are multiple copies of RM(r, m) embedded inside of RM(r, m + k) with the same bit 0, one can utilize two of them separately to compute estimates of bit 0 based on the RM(r, m) code structure. These two looks can be combined to get a better estimate of bit 0. Unless they are perfectly correlated, they will actually provide a strict improvement over one estimate. The interplay between the rate difference, R(r, m) - R(r, m + k), and the two-look phenomenon plays a key role in this work.
Discussion of the code rate after puncturing. The above results show that it is possible to remove (i.e., puncture) half of the code symbols from an RM$(r, m+1)$ code to get an RM$(r, m)$ and this puncturing increases the code rate by less than $\frac{2}{\sqrt{m}}$. The original code has $2^{2m+1}R(r, m+1)$ codewords and the punctured code has $2^{2m}R(r, m)$ codewords. This implies that, on average, roughly

$$\frac{2^{2m+1}R(r, m+1)}{2^{2m}R(r, m)} = 2^{2m}(2R(r, m+1)−R(r, m)) \geq 2^{2m}(R(r, m+1)−2/\sqrt{m})$$

codewords in the original code must collapse onto a single codeword in the punctured code.

For a code, recall that an information set is a subset of code symbols where all possible patterns appear \[29\]. It follows that the number of codewords is unchanged if one punctures a set of symbols that is disjoint from any fixed information set. In our example, if the code rate is less than 1/2, then puncturing such a set must increase the code rate by a factor of 2.

One can also ask whether random linear codes have some of the same properties. For now, we will ignore the fact that such codes have a very small probability of being transitive and focus only on the rate after puncturing. For the random generator matrix model, applying a fixed puncturing pattern to a random generator matrix, with design rate $R$ and length $N$, simply gives a random generator matrix from the same ensemble with length $N'$ and design rate $R' = RN/N'$. Thus, the existence of a puncturing pattern that nearly preserves the rate can be related to the concentration of the code rate around its expected value. For this model, one can show that the probability of the rate being less than $R' - \delta$ is upper bounded by $2^{h(\delta)N−\delta RN^2}$. Since there are at most $2^N$ puncturing patterns, the union bound implies that the probability of any puncturing pattern resulting in rate smaller than $R' - \delta$ is upper bounded by $2^{(h(\delta)+1)N−\delta RN^2}$. Thus, this is extremely unlikely to occur for large $N$.

Still, RM codes are not alone with this property. One can show that certain sequences of multidimensional product codes also have this property. Similarly, it would be interesting to study whether or not there are other algebraic code constructions (e.g., BCH codes) with this property.

It is worth noting that a property related to the above discussion was recently discussed in the context of successive-cancellation list decoding for polar-like codes \[30\]. In that work, it is shown that an affine-invariant code with rate $R$ and length-$2^{m+1}$ can be transformed into a code of length-$2^m$ whose rate $R'$ approaches $R$ for large $m$. When applied to RM codes, the transformation process they consider is a mapping from RM$(r, m+1)$ to RM$(r-1, m)$. Thus, some of the ideas in \[30\] might provide an avenue for extending some of our results to affine-invariant codes.

3 Background

3.1 Binary Memoryless Symmetric Channels

An information channel $W$ is defined by an input alphabet $X$, an output alphabet $Y$, and a transition probability that maps elements of the input alphabet to probability measures on the output alphabet. We follow the convention of representing the transition probability using a density function $w(y|x)$ with respect to a base measure on the output alphabet (e.g., counting measure if output distribution is discrete or Lebesgue measure if $Y = \mathbb{R}^d$ and the output distribution is continuous). In this paper, we restrict our attention to the channels satisfying the following.

Definition 10 (BMS Channel \[31\] p. 178). A channel with binary input alphabet $X = \{\pm 1\}$ and output alphabet $Y = \mathbb{R}$ is said to be symmetric if the transition probability satisfies $w(y|+1) = w(-y|-1)$ for all $y \in Y$. A binary memoryless symmetric (BMS) channel consists of a sequence of channels uses such that:

- Each channel use has binary input alphabet $\{\pm 1\}$ and is symmetric.
- Conditional on the input to the $i$-th channel, the output of the $i$-th channel is independent of all of the other channel uses.

Every channel satisfying Definition 10 can be expressed as a multiplicative noise channel. Specifically, if the input is a random vector $X \in \{\pm 1\}^N$ then the output $Y \in Y^N$ is given by

$$Y = X \odot Z$$

(3)
where \( \odot \) denotes the Hadamard (entrywise) product and \( Z \in \mathbb{Z}^N \) is an independent random vector whose entries are independent with \( Z_i \) drawn according to the distribution of the output in the \( i \)-th channel given the input is \( +1 \) \( \text{[31]} \) p. 182. In many cases, it is also assumed that the \( \{Z_i\} \) are identically distributed, and thus each channel is described by the same transition probability. Two BMS channels are called equivalent if their multiplicative noise representations are the same.

Examples of BMS channels include the following:

- Binary erasure channel (BEC) where \( Z_i \in \{0, 1\} \) transmits the input faithfully if \( Z_i = 1 \) and outputs an erasure if \( Z_i = 0 \).
- Binary symmetric channel (BSC) where \( Z_i \in \{\pm 1\} \) transmits the input faithfully if \( Z_i = 1 \) and flips the input if \( Z_i = -1 \).
- Additive white Gaussian noise (AWGN) channel where \( Z_i \sim \mathcal{N}(0, \sigma_i^2) \) for some noise power \( \sigma_i^2 \).

While the definition of BMS channels can be extended to output alphabets beyond \( \mathbb{R} \) (e.g., which do not satisfy the multiplicative noise decomposition above), it turns out that every BMS channel defined in the more general sense has a real sufficient statistic that satisfies the definition given above. So there is no loss of generality in restricting our attention to channels satisfying [1]. See Appendix A.1 for details.

### 3.2 MMSE and Bit Error Rate

For a real-valued random variable \( X \) and an observation \( Y \), the minimum mean-squared error (MMSE) of \( X \) given \( Y \) is given by

\[
\text{mmse}(X \mid Y) := E \left[ (X - E[X \mid Y])^2 \right] = E [X^2] - E \left[ E[X \mid Y] \right]^2
\]

where the second expression holds whenever \( E[X^2] \) is finite. For a binary random variable \( X \in \{\pm 1\} \) and an observation \( Y \), the bit-error probability of the MAP decision rule is defined by

\[
\text{BER}(X \mid Y) := 1 - E \left[ \max\{\Pr(X = 1 \mid Y), \Pr(X = -1 \mid Y)\} \right],
\]

where we use the convention that \( \Pr(X = 1 \mid Y) \) is a random variable. For digital communication systems with error-correcting codes, the bit-error probability (of codeword bits) after decoding is an important performance metric. The system may be considered reliable if this probability can be made arbitrarily small. A more stringent requirement is that the block-error probability (i.e., the probability that any bit in the codeword is not correct) can be made arbitrarily small. Of course, this is preferable but can be more challenging to analyze.

**Lemma 11.** For a binary random variable \( X \in \{\pm 1\} \) observed as \( Y \), the quantities \( \text{mmse}(X \mid Y) \) and \( \text{BER}(X \mid Y) \) satisfy

\[
\frac{1 - \sqrt{1 - \text{mmse}(X \mid Y)}}{2} \leq \text{BER}(X \mid Y) \leq \frac{1}{2} \text{mmse}(X \mid Y).
\]

Thus, for a sequence of observations of \( X \), the BER converges to zero if and only if the MMSE converges to zero.

**Proof.** See Section 6.1.

For general \( X, Y, \) and \( Y' \), the following lemma bounds the mean-squared difference between conditional mean estimates of \( X \) given the observations \( Y \) and \( Y' \).

**Lemma 12.** Let \( X, Y, \) and \( Y' \) be random variables on a common probability space and assume \( X \) is real-valued. If \( \text{mmse}(X \mid Y) \) and \( \text{mmse}(X \mid Y') \) are both finite, then

\[
E \left[ (E[X \mid Y'] - E[X \mid Y])^2 \right] \leq 2 \text{mmse}(X \mid Y) + 2 \text{mmse}(X \mid Y') - 4 \text{mmse}(X \mid Y, Y').
\]
Consider the case where one gets two observations $Y, Y'$ of the random variable $X$ that are identically distributed in the sense that $(X, Y)$ and $(X, Y')$ are equal in distribution. The above bound shows that, if $\text{mmse}(X|Y)$ is close to $\text{mmse}(X|Y, Y')$, then $\mathbb{E}[X|Y]$ and $\mathbb{E}[X|Y']$ are close in the mean-square sense. Note that there is no assumption that $Y, Y'$ are conditionally independent given $X$. This result plays a key role in the two-look bound described in Section 5.2.2.

3.3 Generalized Extrinsic Information Transfer Functions

The generalized extrinsic information transfer (GEXIT) function [31, 32] provides a powerful tool for the analysis of communication problems. This section briefly reviews some of the main ideas used in the analyses of GEXIT functions as well as some related concepts involving I-MMSE relations.

Rather than focusing on a specific information channel $W$, the main object of interest is a family of channels $\{W(t)\}$ indexed by a real-valued parameter $t \in [0, 1]$, where each $W(t)$ represents a channel from a common input alphabet $\mathcal{X}$ (not necessarily binary) to a common output alphabet $\mathcal{Y}$. For concreteness, it is assumed throughout that $t = 0$ is the perfect channel (i.e., the input is determined uniquely by the output) and $t = 1$ is the uninformative channel (i.e., the output does not depend on the input). For a given number of channel uses $N$, the problem is described as follows:

- The input $X \in \mathcal{X}^N$ is a random vector with distribution $p_X$. For communication systems, this is typically the uniform distribution over a subset of the input space defined by a code.
- The output $Y \in \mathcal{Y}^N$ is an observation of $X$ through a memoryless channel where each $Y_i$ is an observation of $X_i$ through the channel $W(t_i)$ for some $t_i \in [0, 1]$.

In some cases, we use the notation $Y(t_0, \ldots, t_{N-1})$ to make the dependence on the channel parameters explicit. Instead, if all channel parameters take the common value $t$, then we use the notation $Y(t)$.

Once the input distribution and the family of channels have been specified, the high-level idea is to study how certain quantities, such as the entropy and the bit error rate, depend on the underlying channel parameters. Under the assumptions on the channel family outlined above, the conditional entropy of the input given the output satisfies the boundary conditions $H(X | Y(0)) = 0$ (the perfect channel) and $H(X | Y(1)) = H(X)$ (the uninformative channel). If we also assume that the family of channels depends smoothly on the parameter $t$ (e.g., that the mapping $(t_0, \ldots, t_{N-1}) \mapsto H(X | Y(t_0, \ldots, t_{N-1}))$ is differentiable on $[0, 1]^N$) then we can use the fundamental theorem of calculus and the law of the total derivative to obtain the following decomposition:

$$H(X) = \int_0^1 \left\{ \frac{d}{ds} H(X | Y(s)) \right\}_{s=t} dt = \sum_{i=0}^{N-1} \int_0^1 \left\{ \frac{\partial}{\partial s_i} H(X | Y(s_0, \ldots, s_{N-1})) \right\}_{s_0=\ldots=s_{N-1}=t} dt,$$

where in the second line, the partial derivative is taken with respect to parameter in the $i$-th channel.

There are two special cases where the partial derivatives in (4) can be recognized as measures of uncertainty associated with the $i$-th entry of the input:

- **Erasure**: Consider the family of erasure channels where $\mathcal{Y} = \mathcal{X} \cup \{?\}$ and the probability of erasure is equal to $t$. In this case, it is straightforward to show that the $i$-th partial derivative in (4) is equal to $H(X_i | Y_{\sim i}(t))$ where the subscript $\sim i$ means that the $i$-th term is omitted. The mapping described by $t \mapsto H(X_i | Y_{\sim i}(t))$ is called the extrinsic information transfer (EXIT) function and it has found many uses in the literature [33] [34] [35] [7].

- **AWGN**: Consider the family of AWGN channels defined by $\mathcal{X} = \mathcal{Y} = \mathbb{R}$ where each $X_i$ is observed as $Y_i(t_i) = \sqrt{\text{snr}(t_i)} X_i + V_i$ with $V_i$ equal to i.i.d. standard Gaussian noise. We assume that $\text{snr}(t)$ is a non-increasing function of $t$ with $\text{snr}(t) \to \infty$ as $t \to 0$ and $\text{snr}(1) = 0$. In this
case, it follows from the I-MMSE relation \[36\] that the \(i\)-th partial derivative in \[4\] is equal to \(-\frac{1}{2}\ \text{snr}'(t) \ \text{mmse}(X_i \mid Y(t))\) where \(\text{snr}'(t)\) is the derivative of \(\text{snr}(t)\) and \(\text{mmse}(X_i \mid Y(t))\) is the minimum mean-squared error of the \(i\)-th input. This relationship has played a key role in the analysis coding problems as well as high-dimensional inference problems involving Gaussian noise \[37\] \[38\] \[39\] \[40\] \[41\].

Going beyond the BEC and AWGN, the partial derivatives in \[4\] associated with a general channel no longer have such a simple interpretation. Nevertheless, many of the ideas developed in the context of the BEC and AWGN cases are still applicable. Historically, the idea of GEXIT functions is introduced and developed by Méasson, Montanari, Richardson, and Urbanke in \[32\].

**Definition 13** (GEXIT function). Let \(X \in \mathcal{X}^N\) be a random vector with distribution \(p_X\) and let \(Y(t_0, \ldots, t_{N-1}) \in \mathcal{Y}^N\) be an observation of \(X\) through a memoryless channel where each \(Y_i\) is an observation of \(X_i\) through the channel \(W(t_i)\). The GEXIT function for entry \(i \in [N]\) is defined to be the partial derivative w.r.t. the channel parameter for the \(i\)-th output:

\[
G_i(t) := \frac{\partial}{\partial s_i} H(X \mid Y(s_0, \ldots, s_{N-1})) \bigg|_{s_0=\cdots=s_{N-1}=t}, \quad t \in [0, 1].
\]

The full power of GEXIT analysis is realized when the input distribution and the channel satisfy certain symmetry properties that imply the GEXIT functions are all identical, i.e., \(G_0 = \cdots = G_{N-1}\). In this case, \(4\) implies that, for all \(i \in [N]\), we have

\[
\int_0^1 G_i(t) \, dt = \frac{1}{N} H(X).
\]

For the BEC and AWGN channel, this result connects a well-known reliability measure associated with a single entry element of \(X\) to the entropy of the entire vector \(X\). A sufficient condition under which the GEXIT functions are identical is that the distribution of the input vector has transitive symmetry.

**Definition 14** (Symmetry and Transitivity). Let \(S_N\) be the set of permutations (i.e., bijective functions) mapping \([N]\) to itself. The symmetry group of a random vector \(X = (X_0, \ldots, X_{N-1})\) is defined to be

\[
\mathcal{G} := \{ \pi \in S_N : X_\pi \overset{d}{=} X \},
\]

where \(X_\pi := (X_{\pi(0)}, \ldots, X_{\pi(N-1)})\) and \(\overset{d}{=}\) indicates equality in distribution. We say that \(X\) has transitive symmetry if \(\mathcal{G}\) is transitive (i.e., for all \(i, j \in [N]\), there is a \(\pi \in \mathcal{G}\) such that \(\pi(i) = j\)). We say that \(X\) has a doubly-transitive symmetry if \(\mathcal{G}\) is doubly transitive (i.e., for distinct \(i, j, k \in [N]\), there is a \(\pi \in \mathcal{G}\) such that \(\pi(i) = i\), and \(\pi(j) = k\).

### 4 Preliminary Results

#### 4.1 BMS Families Ordered by Degradation

Our approach builds upon the GEXIT analysis outlined in Section \[3.3\]. Rather than focusing on a particular BMS channel we study a family of BMS indexed by a parameter \(t \in [0, 1]\) where \(t = 0\) is the perfect channel and \(t = 1\) is the uninformative channel. We also require that this family is ordered with respect to degradation in the sense that \(W(t)\) is degraded with respect to \(W(s)\) for all \(0 \leq s \leq t \leq 1\).

Equivalently, for any distribution on the input \(X\) there exists a joint distribution on \((X, Y(s), Y(t))\) such that

- \(Y(s)\) is an observation of \(X\) through channel \(W(s)\)
- \(Y(t)\) is an observation of \(X\) through channel \(W(t)\)
- \(X - Y(s) - Y(t)\) forms a Markov chain.
Figure 3: Joint range of entropy and MMSE functions associated with a family of BMS channels as described by (7). The upper boundary is attained by the BSC and the lower boundary is attained by the BEC [42]. Also, the derivative ratio $H'(t)/M'(t)$ is bounded from below by $1/(2 \ln 2)$.

We remark that this degradation assumption is also standard in the literature on GEXIT analysis [31]. See Appendix A.2 for a precise definition of channel degradation and some of its consequences.

There are a few well-known examples of channel families that are ordered by degradation. Some examples are the family of BECs where the erasure probability transitions from 0 to 1, the family of BSCs where the crossover probability transitions from 0 to $1/2$, and the family of AWGN channels where the noise power transitions from 0 to $+\infty$.

The Shannon capacity of a BMS channel is equal to the mutual information between the input and output when the input is uniformly distributed [31]. For a family of BMS channels, two important metrics are provided by the entropy and the MMSE for a uniform input distribution:

**Definition 15.** Let $\{W(t)\}_{t \in [0,1]}$ be a family of BMS channels that is ordered w.r.t. degradation where $W(0)$ is the perfect channel and $W(1)$ is the uninformative channel. The entropy function $H: [0,1] \to [0,1]$ and MMSE function $M: [0,1] \to [0,1]$ are defined according to

$$H(t) := H(X^* \mid Y^*(t))$$

$$M(t) := \mathbb{E} \left[ (X^* - \mathbb{E}[X^* \mid Y^*(t)])^2 \right]$$

where $X^*$ is uniformly distributed on $\{\pm 1\}$ and $Y^*(t)$ is an observation of $X^*$ through the channel $W(t)$. The family is said to be absolutely continuous if the entropy function is absolutely continuous, i.e., if there exists a function $H': [0,1] \to \mathbb{R}$ such that $H(b) - H(a) = \int_a^b H'(t) \, dt$ for all $a, b \in [0,1]$.

The entropy and MMSE functions have a number of important functional properties. Under the assumed degradation, both functions are non-decreasing with $H(0) = M(0) = 0$ and $H(1) = M(1) = 1$. For each $t \in [0,1]$, the Shannon capacity of the channel $W(t)$ is equal to $C(t) = 1 - H(t)$. It is known that the extremal relationships between the entropy and the MMSE are attained by the BEC and BSC channels [42]. For example,

$$M(t) \leq H(t) \leq h_b \left( \frac{1 - \sqrt{1 - M(t)}}{2} \right)$$

where $h_b(x) := -x \log_2(x) - (1 - x) \log_2(1 - x)$ is the binary entropy function. Equality on the left is attained by the BEC and equality on the right is attained by the BSC. This type of phenomenon is also known to be somewhat typical [43]. Another property, perhaps less known, is that the difference in
entropy can be used to upper bound the difference in MMSE:

\[ \mathcal{M}(t) - \mathcal{M}(s) \leq 2 \ln(2) (H(t) - H(s)) , \quad 0 \leq s \leq t \leq 1. \quad (8) \]

The proof of (8) follows from applying Lemma 16 to \( H(t) - H(s) \) and keeping only the first term in the expansion. In the case of the BSC, it can be verified that the factor \( 2 \ln(2) \) is tight in the limit where the crossover probability approaches \( 1/2 \) (see Figure 3). Also, as discussed in Lemma 38, this inequality is sufficient for the absolute continuity of the entropy function to imply the absolute continuity of the MMSE function.

4.2 GEXIT and I-MMSE Properties

In this section, we present a number of useful results that characterize GEXIT functions for binary-input channels. GEXIT functions were introduced in [31, p. 196] and analyzed further by a variety of authors [15, 32, 40, 47]. Our treatment is based solely on the moments of the conditional expectation.

Lemma 16 (Entropy Expansion [48, 49, 42]). If \( Y \) is an observation of a random variable \( X \in \{ \pm 1 \} \), then

\[ H(X | Y) = \frac{1}{\ln 2} \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \left( 1 - \mathbb{E} \left[ \mathbb{E} \left[ X | Y \right]^{2k} \right] \right). \]

Furthermore, if \( Y' \) is an observation of a random variable \( X' \in \{ \pm 1 \} \) such that \((X',Y')\) is independent of \((X,Y)\), then the entropy of the product \( X \cdot X' \) satisfies

\[ H(X \cdot X' | Y, Y') = \frac{1}{\ln 2} \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \left( 1 - \mathbb{E} \left[ \mathbb{E} \left[ X | Y \right]^{2k} \right] \mathbb{E} \left[ \mathbb{E} \left[ X' | Y' \right]^{2k} \right] \right). \]

Proof. See Section 6.2.

The next result essentially follows from the entropy expansion in Lemma 16 and the duality rule for entropy used in the density evolution analysis of low-density parity-check codes [31, p. 196]. The difference is that our approach does not require \( X \) to be uniform. Thus, we provide a short self-contained proof.

Lemma 17 (Two-look Formula). Let \( X \in \{ \pm 1 \} \) be a random variable transmitted through two independent binary-input channels \( W_{Y|X} \) and \( W_{Z|X} \) whose outputs are \( Y \) and \( Z \), respectively. Further, suppose that \( W_{Y|X} \) is symmetric and let \((X^*,Y^*)\) be an independent input-output pair from this channel where \( X^* \) is uniformly distributed on \( \{ \pm 1 \} \). Then, we have

\[ H(X | Y, Z) = \frac{1}{\ln 2} \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \left( 1 - \mathbb{E} \left[ \mathbb{E} \left[ X^* | Y^* \right]^{2k} \right] \right) \left( 1 - \mathbb{E} \left[ \mathbb{E} \left[ X | Z \right]^{2k} \right] \right). \quad (9) \]

Proof. See Section 6.2.

By applying the two-look formula to combine a single look with a non-uniform prior, we obtain a general formula for \( H(X|Y) \) that separates the contribution of the channel and the input distribution. Specifically, if we assume \( \mu = \mathbb{E}[X|Z] = (1 - 2p) \) is almost surely constant (corresponding to \( X = -1 \) having prior probability \( p \) and \( Z \) being uninformative), then the direct application of (9) leads to

\[ H(X | Y) = H(X | Y, Z) = \frac{1}{\ln 2} \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \left( 1 - \mathbb{E} \left[ \mathbb{E} \left[ X^* | Y^* \right]^{2k} \right] \right) \left( 1 - \mu^{2k} \right). \]

If \( X \) is equiprobable and \( W_{Z|X} \) is a BSC with error probability \( p = \frac{1-\mu}{2} \), then \( \mathbb{E}[X|Z]^{2k} = (1-2p)^{2k} = \mu^{2k} \). It follows that one gets exactly the same formula for the entropy in this case.
Lemma 18. For a family of BMS channels ordered by degradation that is absolutely continuous according to Definition \([15]\), let \(X^*\) be a random variable uniformly distributed on \([\pm 1]\) and let \(Y^*(t)\) be an observation through the channel with parameter \(t \in [0,1]\). Let \(Z\) be an observation of \(X\) through a BSC with error probability \(p = \frac{1-\mu}{2}\). Then, \(H(X^*|Y^*(t), Z) = \mathcal{H}_\mu(t)\) where

\[
\mathcal{H}_\mu(t) := \frac{1}{\ln 2} \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \left( 1 - \mathbb{E}[X^* | Y^*(t)]^{2k} \right) (1 - \mu^{2k})
\]

satisfies \(\mathcal{H}_0(t) = H(X^* | Y^*(t)) = \mathcal{H}(t)\) and \(\mathcal{H}_\mu(1) = h_\mu(\frac{1-\mu}{2})\). The function \(\mathcal{H}_\mu(t)\) is non-decreasing and absolutely continuous in \(t\), and non-increasing in \(\mu^2\). Its derivative, \(\mathcal{H}'(t)\), exists for almost all \(t \in [0,1]\) and is non-increasing in \(\mu^2\).

Proof. The formula follows from the fact that \(\mathbb{E}[\mathbb{E}[Z|X]^{2k}] = \mu^{2k}\) when \(W_{Z|X}\) is a BSC. Lemma 16 shows that, for \(\mu = 0\), that it equals \(\mathcal{H}(t)\) and, for \(\mu = 1\), it equals 0. The function \(\mathcal{H}_\mu(t)\) is non-decreasing in \(t\) (due to the degradation ordering of channel) and non-increasing in \(\mu^2\) (from the given formula).

In addition, one can factor out \(1 - \mu^{2k}\) to see that, for \(0 \leq t \leq t' \leq 1\), the difference \(\mathcal{H}_\mu(t') - \mathcal{H}_\mu(t)\) is non-increasing in \(\mu^2\). Thus, for all \(\mu \in [-1,1]\), this difference is upper bounded by the value for \(\mu = 0\), which is \(\mathcal{H}(t') - \mathcal{H}(t)\). From this, we see that \(\mathcal{H}_\mu(t)\) is absolutely continuous, differentiable for almost all \(t \in [0,1]\), and its derivative is non-decreasing in \(\mu^2\). \(\square\)

The next result is a further implication of the two-look formula that bounds one’s ability to estimate a binary variable from two observations subject to an MMSE lower bound on one of the observations.

Lemma 19. Let \(X\) be uniform on \([\pm 1]\) and let \(Y\) and \(Z\) be conditionally independent observations such that the \(W_{Z|X}\) channel is symmetric and satisfies \(\text{mmse}(X|Z) \geq 1 - \epsilon\) for some \(0 \leq \epsilon \leq 1\). Then, the following bounds hold:

\[
H(X | Y, Z) \geq (1 - \epsilon)H(X | Y)
\]

\[
\text{mmse}(X | Y, Z) \geq \text{mmse}(X | Y) - 2\ln(2)\epsilon
\]

\[
\text{BER}(X | Y, Z) \geq \text{BER}(X | Y) - \sqrt{\frac{\ln(2)\epsilon}{2}}.
\]

Proof. See Section 6.2.

Lemma 20 (I-MMSE relations for BMS channels). For a family of BMS channels ordered by degradation that is absolutely continuous according to Definition \([15]\) let \(X \in \{\pm 1\}\) be a random variable and let \(Y(t)\) be an observation through the channel with parameter \(t \in [0,1]\). Let \((Z, Z_1, Z_2)\) be conditionally independent observations of \(X\) where \(Z_2\) is degraded w.r.t. \(Z_1\). Then, \(H(X|Y(t), Z)\) is absolutely continuous on \([0,1]\). Thus, for almost all \(t \in [0,1]\) (e.g., where all of the derivatives exist), we have

\[
\frac{d}{dt} H(X|Y(t), Z) \leq \frac{\text{mmse}(X|Z) - 1}{2\ln 2} \mathcal{H}'(t),
\]

and

\[
\frac{d}{dt} H(X|Y(t), Z) \leq \mathcal{H}'(t) \sqrt{1 - \text{mmse}(X|Z)(t)},
\]

and

\[
\frac{d}{dt} (H(X|Y(t), Z_2) - H(X|Y(t), Z_1)) \geq \frac{1}{2\ln 2} \mathcal{M}'(t)(\text{mmse}(X|Z_2) - \text{mmse}(X|Z_1)).
\]

Proof. See Section 6.2.

Now, we use Lemma 20 to establish and bound the GEXIT function under the assumption that \(\mathcal{H}(t)\) is absolutely continuous.
Lemma 21. Let \( X \in \{\pm 1\}^N \) be a random vector and \( Y_0(s_0), \ldots, Y_{N-1}(s_{N-1}) \) be its observation through a family of BMS channels ordered by degradation that is absolutely continuous according to Definition \( 13 \). Then, \( s_i \mapsto H(X \mid Y_i(s_i), Y_{-i}(t)) \) is absolutely continuous on \([0,1]\) for all \( t \in [0,1] \) and \( G_i(t) \) exists almost everywhere on \([0,1]\). Let \( M_i(t) := \text{mmse}(X \mid Y_i(t)) \) denote the extrinsic MMSE of \( X_i \) given \( Y(t) \). Then, whenever all the derivatives exist, we find that

\[
M_i(t)\mathcal{H}'(t) \leq G_i(t) \leq \frac{(M_i(t) - 1)\mathcal{M}'(t)}{2 \ln 2} + \mathcal{H}'(t),
\]

(13)

\[
G_i(t) \leq \mathcal{H}' \sqrt{1 - M_i(t)}(t).
\]

(14)

Proof. See Section 6.2.

Remark 22. In comparison to \([31, 32]\), our GEXIT formulation is somewhat more general and requires fewer regularity assumptions. In particular, we allow the channel family to be parameterized arbitrarily and we show that the GEXIT function exists as long as its entropy function \( \mathcal{H}(t) \) is absolutely continuous. An alternative approach to the analysis of GEXIT functions, which shares some of these properties, can be found in \([36]\).

4.3 Linear Codes on BMS Channels

A set \( C \subseteq \mathbb{F}_2^N \) defines a binary linear code (i.e., a subspace of \( \mathbb{F}_2^N \)) if and only if it is closed under addition, that is to say \( u \oplus u' \in C \) for all \( u, u' \in C \), where \( \oplus \) represents element-wise modulo-2 addition. To transmit a message over a binary channel, each codeword \( u \in C \) is mapped to a channel input sequence in \( \{\pm 1\}^N \) via the mapping \( u \mapsto (-1)^u \). The resulting set of channel input sequences is denoted by \( C_x \).

A remarkable property of linear codes on BMS channels is that many performance metrics do not depend on the transmitted codeword \([31\text{ p. 190}]\). This property greatly simplifies the analysis of coding problems because it means that one may condition on the event that the all ones input is transmitted (corresponding to the all zeros linear codeword). Note that under this event, the outputs of the BMS channel are independent random variables.

Next, we describe a consequence of linearity and channel symmetry that is useful for our analysis.

Lemma 23. Let \( X \) be distributed uniformly on the set of channel input sequences \( C_x \subseteq \{\pm 1\}^N \) associated with the binary linear code \( C \subseteq \mathbb{F}_2^N \) and let \( Y \) be an observation of \( X \) through a BMS channel of the form \( Y = X \odot Z \) where \( Z \in \mathbb{Z}^N \) is an independent vector with independent entries. For \( i \in [N] \) and \( S \subseteq [N] \), define \( f(y_S) := \mathbb{E}[X_i \mid Y_S = y_S] \) to be the conditional expectation of the \( i \)-th input given the outputs indexed by \( S \). Then, for all \( x \in C_x \) and \( y_S \) in the support of \( Y_S \) the following identity holds:

\[
f(y_S) = x_i f_S(x_S \odot y_S).
\]

In particular, this implies that

\[
f(Y_S) = X_i f(Z_S).
\]

Proof. By Bayes rule, the conditional probability mass function of \( X \) given \( Y_S = y_S \) satisfies

\[
p_{X \mid Y_S}(x \mid y_S) \propto p_X(x) \prod_{i \in S} w(y_i \mid x_i), \quad x \in \{\pm 1\}^N
\]

where \( p_X(x) \) is the uniform distribution over the codewords, \( w(y \mid x) \) is the transition probability, and the unspecified constant of proportionality is chosen to ensure the function sums to one. The fact that a linear code is closed under addition means that it is a subgroup of \( \mathbb{F}_2^N \), and thus for any \( u \in C \), a vector \( u' \in \mathbb{F}_2^N \) satisfies \( u' \in C \) if and only if \( u \oplus u' \in C \). Using the code to input mapping \( x = (-1)^u \), this implies that for any \( x \in C \), a vector \( x' \in \{\pm 1\}^N \) satisfies \( x' \in C_x \) if and only if \( x \odot x' \in C_x \).

To proceed, fix any \( x' \in C_x \) and observe that \( p(x) = p(x' \odot x) \) for all \( x \in \{\pm 1\}^N \). Meanwhile, the assumption of channel symmetry means that \( w(y_S \mid x_S) = w(x'_S \odot y_S \mid x'_S \odot x_S) \) for all \( x \in \{\pm 1\}^N \). Together, these statements imply that

\[
p_{X \mid Y_S}(x \mid y_S) = p_X(x' \odot x \mid x'_S \odot y_S), \quad x \in \{\pm 1\}^N.
\]
In view of this identity, the conditional expectation satisfies
\[
    f_S(y_S) = \sum_{x \in \{\pm 1\}^N} x_i p_{X \mid Y_S}(x \mid y_S)
    = \sum_{x \in \{\pm 1\}^N} x_i p_{X \mid Y_S}(x' \odot x \mid x'_S \odot y_S)
    = x'_i \sum_{x \in \{\pm 1\}^N} (x'_i x_i) p_{X \mid Y_S}(x' \odot x \mid x'_S \odot y_S)
    = x'_i f_S(x'_S \odot y_S).
\]

The final statement follows from noting that \( X \odot Y = Z \).

The approach used below is essentially the same as conditioning on the transmission of a particular codeword. However, for conceptual reasons, we find it convenient to frame some results in terms of two coupled channel outputs that are conditionally independent given the input. Our next result is framed this way and provides an identity for the MMSE associated with estimating a single input.

**Lemma 24.** Let \( X \) be distributed uniformly on the set of channel input sequences \( C_x \subseteq \{\pm 1\}^N \) associated with the binary linear code \( C \subseteq \mathbb{F}_2^N \) and let \( Y \) be an observation of \( X \) through a BMS channel of the form \( Y = X \odot Z \) where \( Z \in \mathbb{Z}_N^N \) is an independent vector with independent entries. For each \( i \in [N] \) and \( S \subseteq [N] \), the following identity holds
\[
    \text{mmse}(X_i \mid Y_S)(1 - \text{mmse}(X_i \mid Y_S)) = \frac{1}{2} \mathbb{E} \left[ \left( \mathbb{E} [X_i \mid Y_S] - \mathbb{E} [X_i \mid Y'_S] \right)^2 \right],
\]

where \( Y' = X \odot Z' \) is an independent second use of the channel with the same input \( X \). Furthermore, for every partition \( (B_1, \ldots, B_K) \) of \( S \), we have the upper bound
\[
    \text{mmse}(X_i \mid Y_S)(1 - \text{mmse}(X_i \mid Y_S)) \leq \frac{1}{2} \sum_{k=1}^K \mathbb{E} \left[ \left( \mathbb{E} [X_i \mid Y_S] - \mathbb{E} [X_i \mid Y_{S}^{B_k}] \right)^2 \right],
\]

where \( Y_{S}^{B_k} \) is a modified observation of \( Y_S \) where the entries indexed by \( B_k \) are resampled independently according to the same input \( X \).

**Proof.** Define the conditional expectation \( f(y_S) := \mathbb{E} [X_i \mid Y_S = y_S] \) and observe that
\[
    \mathbb{E} [X_i f(Y_S) \mid Y_S] = \mathbb{E} \left[ \mathbb{E} [X_i \mid Y_S] \mathbb{E} [X_i \mid Y_S] \mid Y_S \right] = f^2(Y_S)
\]

almost surely where the equalities follow from nested conditional expectation. This implies that
\[
    \text{Var}(X_i f(Y_S)) = \mathbb{E} \left[ X_i^2 f^2(Y_S) \right] - \mathbb{E} [X_i f(Y_S)]^2
    = \mathbb{E} \left[ f^2(Y_S) \right] - \mathbb{E} [f^2(Y_S)]^2
    = (1 - \mathbb{E} [f^2(Y_S)] \mathbb{E} [f^2(Y_S)])
    = \text{mmse}(X_i \mid Y_S)(1 - \text{mmse}(X_i \mid Y_S)),
\]

where the simplifications are based on \( X_i^2 = 1 \) and \( \text{mmse}(X_i \mid Y_S) = 1 - \mathbb{E} [f^2(Y_S)] \). This decomposition holds generally for any random variable \( X_i \in \{-1, 1\} \) and any channel \( X_i \rightarrow Y_S \).

Next, we appeal to the special properties of the BMS channel and the linear code. Specifically, by Lemma 23 it follows that \( X_i f(Y_S) = f(Z_S) \). Writing \( Y = X \odot Z \) and \( Y' = X \odot Z' \) where \( Z' \) is an independent copy of \( Z \), we can now write
\[
    \text{Var}(X_i f(Y_S)) = \text{Var}(f(Z_S))
    = \frac{1}{2} \mathbb{E} \left[ (f(Z_S) - f(Z'_S))^2 \right]
\]
where the second line can be verified by expanding the square, the third line holds since $X_i^2 = 1$, and the last step is another application of Lemma 23. Combining the two difference expressions for $\text{Var}(X_i f(Y_S))$ gives the desired identity.

To prove the upper bound, recall that $f(Z)$ is a bounded function of independent random variables. Hence, we can apply the Efron-Stein inequality [50, Theorem 3.1] with respect to the partition $(Z_{B_1}, \ldots, Z_{B_k})$ to conclude that

$$\text{Var}(f(Z)) \leq \sum_{k=1}^K \frac{1}{2} \mathbb{E} \left[ \left( f(Z_{S_k}) - f(Z_{S_k}^B) \right)^2 \right],$$

where $Z_{S_k}^B$ denotes the version of $Z_S$ with the entries indexed by $B_k$. Multiplying the terms in the square by $X_i$ and then applying Lemma 23 leads to the stated bound, which is given in terms of $Y_S$ and $Y_S^B$.

Finally, we need the following result concerning the distribution of a pair of estimates based on correlated observations.

**Lemma 25.** Let $X$ be distributed uniformly on the set of channel input sequences $\mathcal{C}_x \subseteq \{\pm 1\}^N$ associated with the binary linear code $C \subseteq \mathbb{F}_2^N$ and let $Y$ be an observation of $X$ through a BMS channel of the form $Y = X \odot Z$ where $Z \in \mathbb{Z}^N$ is an independent vector with independent and identically distributed entries. Suppose that for $i \in [N]$ there exist disjoint sets $A, B, C \in [N]\{i\}$ and a permutation matrix $\Pi$ such that $(X_i, X_A, X_B)$ is equal in distribution to $(X_i, X_A, \Pi X_C)$. Then,

$$(\mathbb{E}[X_i | Y_A, Y_B], \mathbb{E}[X_i | Y_A, Y_B]) \overset{d}{=} (\mathbb{E}[X_i | Y_A, Y_B], \mathbb{E}[X_i | Y_A, Y_B]),$$

where $Y'$ is an independent second use of the channel with the same input $X$.

**Proof.** Let us define the conditional expectations,

$$f(u, v) := \mathbb{E}[X_i | Y_A = u, Y_B = v],$$

$$g(u, v) := \mathbb{E}[X_i | Y_A = u, Y_C = v].$$

The assumption that $(X_i, X_A, X_B)$ is equal in distribution to $(X_i, X_A, \Pi X_C)$ combined with the assumptions on the channel imply that $(X_i, Y_A, Y_B)$ is equal in distribution to $(X_i, Y_A, \Pi Y_C)$, and thus

$$f(u, v) = g(u, \Pi v). \tag{15}$$

From the channel assumptions, the channel outputs can be expressed as $Y = X \odot Z$ and $Y' = X \odot Z'$ where $X, Z, Z'$ are independent. By Lemma 23 and the relation between $f$ and $g$ in (15), we can write

$$(f(Y_A, Y_B), f(Y_A, Y_B')) = X_i( f(Z_A, Z_B), f(Z_A, Z_B'))
= X_i( f(Z_A, Z_B), g(Z_A, \Pi Z_B'))
\overset{d}{=} X_i( f(Z_A, Z_B), g(Z_A, Z_C))
= (f(Y_A, Y_B), g(Y_A, Y_C)),$$

where the equality in distribution holds because of the assumption that the entries of $Z$ and $Z'$ are independent and identically distributed. \qed
5 Proof of Main Result

We prove that RM codes achieve capacity for any BMS channel in the limit of large blocklength. The first step of our proof is to embed the BMS channel of interest into a family of absolutely continuous BMS channels as described in Definition 15. To further simplify our analysis, we will add the additional assumption that the family of BMS channels is parameterized such that the MMSE function $\mathcal{M}(t)$ defined in (6) is given by $\mathcal{M}(t) = t$.

We emphasize that these assumptions are not restrictive in the sense that, for any BMS channel, there exists a family of channels satisfying these constraints. An explicit construction based on linear interpolation with erasure channels is described in Section 5.4. For the convenience of the reader we restate the channel assumptions for our main result as follows:

**Assumption 1.** We have a family of BMS channels indexed by parameter $t \in [0, 1]$ satisfying the following properties:

A) For a random input $X \in \{-1, 1\}^N$, the output associated with $t \in [0, 1]$ is given by a BMS channel of the form

$$Y(t) = X \odot Z(t)$$

where $Z(t) = \{Z_i(t)\}_{i \in [N]}$ is a random vector, independent of $X$, whose entries are independently and identically distributed according to a probability measure indexed by $t$.

B) The family of BMS channels is ordered with respect to degradation where $t = 0$ is the perfect channel and $t = 1$ is the uninformative channel.

C) The entropy function $\mathcal{H}(t)$ defined in (5) is absolutely continuous on $[0, 1]$.

D) The MMSE function $\mathcal{M}(t)$ defined in (6) satisfies $\mathcal{M}(t) = t$.

While our main result concerns sequences of RM codes of increasing blocklength, many of the steps in our proof hold for a larger class of codes. To make these distinctions apparent, we list here the weaker properties that are sometimes used. We note that all of these properties are satisfied by the RM($r,m$) code with $N = 2^m$. In particular, we always assume that the random input vector $X \in \{-1, 1\}^N$ is distributed uniformly on the channel input sequences of a binary code. In some cases, we also require that:

- the code is linear,
- the code has a transitive symmetry group,
- the code has a doubly transitive symmetry group.

5.1 The Extrinsic MMSE Function

As discussed in Section 3.3, the entropy decomposition in (4) plays an important role in the analysis of the BEC and the AWGN channels, where the partial derivatives provide natural measures of the performance of estimating a single entry of the input. However, one difficulty that arises in extending these approaches to general channels is that the GEXIT function does not seem to have an obvious estimation-theoretic interpretation. The approach taken in this paper is to study a surrogate for the GEXIT function, which we call the extrinsic MMSE function:

**Definition 26 (Extrinsic MMSE function).** Let $X \in \{-1, 1\}^N$ be a random vector and let $Y(t)$ be an observation of $X$ through a BMS channel with parameter $t \in [0, 1]$. The extrinsic MMSE function for input $i \in [N]$ is defined to be

$$M_i(t) := \mathbb{E} \left[ \left( X_i - \mathbb{E}[X_i | Y_{\sim i}(t)] \right)^2 \right], \quad t \in [0, 1].$$
The extrinsic MMSE is similar to the EXIT function $H(X_i \mid Y_{\sim i}(t))$ in the sense that it provides a measure of the ability to estimate the $i$-th input based on the outputs from the other channels. As was the case for the GEXIT function, the extrinsic MMSE is identical for all $i \in [N]$ whenever the input distribution has a transitive symmetry group. In that case, we will sometimes drop the subscript $i$ and denote the extrinsic MMSE by $M(t)$.

For the purposes of proving our main capacity result with respect to the bit error rate, the code is transitive and the extrinsic MMSE has the property that $M(t)$ converges to zero (for a particular sequence of problems of increasing dimension) if and only if the bit error rate converges to zero. To prove that a code sequence, with rate converging to $R$, achieves capacity on the BMS channel family $W(t)$, it is sufficient to show that $M(t)$ converges to zero for all $t \in [0,1]$ such that the code rate $R$ is strictly less than Shannon capacity $C(t)$.

Our proof that RM codes achieve capacity on BMS channels consists of the following steps:

1. **Sharp threshold property**: Show that, for every sequence of RM codes with increasing block-length, the extrinsic MMSE has a sharp threshold property with respect to $t$. Specifically, we show that

$$\int_0^1 M(t)(1 - M(t)) \, dt = O\left(\frac{\ln m}{\sqrt{m}}\right),$$

which implies that $M(t)$ cannot be too different from a step function that jumps from 0 to 1. By itself, this does not imply convergence though because the location where the function jumps is not controlled.

2. **Area theorem**: Show that, if the sequence of RM codes has limiting rate $R$, then the location of the jump in the step function must converge to the unique value of $t$ such that the Shannon capacity $C(t)$ of the BMS channel is equal to the code rate $R$.

The two-step approach of first establishing a sharp threshold and then using an area theorem to localize the jump is now somewhat standard [21,40]. The main novelty in our approach is the reliance on the extrinsic MMSE instead of the GEXIT curve and the mechanism by which we establish convergence to a step function.

### 5.2 Are Two Looks Better Than One?

This section establishes the sharp threshold property for the extrinsic MMSE as described by (16). Observe that, for any input distribution, $M_i(t)$ is non-decreasing in $t$ for each $i \in [N]$ because of the assumed channel degradation. Hence, to show that $M_i(t)$ is close to a 0-1 step function it is sufficient to show that $M_i(t)(1 - M_i(t))$ is close to zero for most (but importantly not all) values of $t$ in the unit interval. Now, we appeal to Lemma 24 which shows that, if the input is defined by a binary linear code, then the following identity holds:

$$M_i(t)(1 - M_i(t)) = \frac{1}{2} \mathbb{E} \left[ (\mathbb{E} [X_i \mid Y_{\sim i}(t)] - \mathbb{E} [X_i \mid Y'_{\sim i}(t)])^2 \right],$$

where $Y'(t)$ is an independent second use of the channel with the same input $X$.

In view of (17) the entire problem of establishing a sharp threshold can be boiled down to the following question:

Assuming that $Y_i$ is not observed, are two independent observations of the remaining code symbols likely to provide significantly different posterior estimates of $X_i$?

In the setting where $X_i$ can be recovered accurately from $Y_{\sim i}$ the answer to this question is clearly negative. Conversely, in the setting where the first look is uninformative (i.e., with high probability the conditional distribution $X_i$ given $Y_{\sim i}$ is close to the prior distribution on $X_i$) then it is unlikely that a second look will make much of a difference. The interesting setting occurs when a single look provides partial information about $X_i$, and so two looks are then better than one in a meaningful sense. Our goal is to show that, w.r.t. the parameter $t$, this “interesting” regime has measure tending to zero, that is to say most values of $t$ are “uninteresting”. Combined with (17) and monotonicity of $M_i(t)$, it follows that $M_i(t)$ converges to a 0-1 step function.
5.2.1 Decomposition of Variance

We consider a decomposition of the variance term appearing in (17) with respect to a set \( B \subset [N]\backslash\{i\} \), which will be specified later. For \( S \subseteq [N] \), define

\[
\Delta^S(t) = \frac{1}{2} \mathbb{E} \left[ \left( \mathbb{E} \left[ X_i \mid Y_{-i}(t) \right] - \mathbb{E} \left[ X_i \mid Y_{-i}(t) \right] \right)^2 \right]
\]

where \( Y^S(t) \) is a modified version of \( Y(t) \) in which the entries indexed by \( S \) have been resampled according to the same input \( X \). If the input distribution is defined by a binary linear code, then we can apply the upper bound in Lemma 24 to the partition given by \( B \) and the singletons in \( A := [N] \backslash (B \cup \{i\}) \) to obtain

\[
M_i(t)(1 - M_i(t)) \leq \Delta^B_i(t) + \sum_{j \in A} \Delta^j_i(t).
\]

We remark that (19) is general in the sense that it holds for every linear code. Moreover, the derivation is based on elementary arguments. For example, starting with (18), the inequality in (19) is simply an application of the Efron-Stein inequality.

In the following, we will bound each term in (19) by first relating it to a GEXIT function, using the results in Section 4.2, and then combining properties of the GEXIT function with some other arguments to bound the integral with respect to \( t \) over the unit interval:

- The term \( \Delta^B_i(t) \) is addressed in Section 5.2.2 where it is shown that if \( N = 2^m \) and the input distribution is uniform on the codewords of the RM\((r, m)\) code, then for all integers \( k \leq m \), there exists for each \( i \in [N] \) a set \( B \subset [N]\backslash \{i\} \) of size \( 2^m - 2^{m-k} - 1 \) such that

\[
\int_0^1 \Delta^B_i(t) \, dt \leq 4 \ln(2) \left( R(r, m) - R(r, m + k) \right),
\]

where we recall that \( R(r, m) \) is the rate of the RM\((r, m)\) code.

- The term \( \Delta^j_i(t) \) is addressed in Section 5.2.3 where it is shown that if the input distribution has a doubly transitive symmetry group, then the following bound holds for all \( i \neq j \),

\[
\int_0^1 \Delta^j_i(t) \, dt \leq \frac{4 \ln 2}{N - 1}.
\]

Combining these results leads to a family of upper bounds on the integral of (19) that is parametrized by \( k \in \{0, \ldots, m\} \). This parameter provides a trade-off between the two terms in the bound. For large values of \( k \), the bound is dominated by the difference between the rates in (20). Conversely, for small values of \( k \), the bound is dominated by the number of singletons not in \( B \), which is given by \( 2^m - k \). Optimizing over the choice of the integer \( k \) gives the following result:

**Lemma 27.** Consider a family of RMS channels satisfying Assumption 7. If the input distribution is uniform on the codewords of the RM\((r, m)\) code, then the extrinsic MMSE satisfies

\[
\int_0^1 M(t)(1 - M(t)) \, dt \leq \rho(m) := \frac{6 \ln(2m) + 34}{5\sqrt{m}}.
\]

**Proof.** For every integer \( k \in [m] \), the bounds in (19), (20), and (21) give

\[
\int_0^1 M(t)(1 - M(t)) \, dt \leq 4 \ln(2) \left( \frac{2^{m-k} - 1}{2^m - 1} + R(r, m) - R(r, m + k) \right)
\]

\[
\leq 4 \ln(2) \left( 2^{-k} + \frac{3k + 4}{5\sqrt{m}} \right)
\]

where the second step follows from the basic inequality \( 2^{m-k} - 1 \leq (2^m - 1)2^{-k} \) and Lemma 7. Next, we choose \( k = \left\lceil \frac{1}{2} \log_2 m \right\rceil \) and note that

\[
2^{-k} + \frac{3k + 4}{5\sqrt{m}} \leq \frac{1}{\sqrt{m}} + \frac{3(\frac{1}{2} \log_2(m) + 1) + 4}{5\sqrt{m}} = \frac{3\log_2(m) + 24}{10\sqrt{m}}.
\]

The final result follows from multiplying this by \( 4 \ln 2 \) and noting that \( 48 \ln(2) \leq 34 \). \( \square \)
5.2.2 Two-Look Bound

This section proves an upper bound on the integral of the term $\Delta^B(t)$ defined in (18) where $B \subset [N] \setminus \{i\}$ is a carefully chosen set. This term can be expressed as

$$\Delta^B(t) := \frac{1}{2} E \left[ (E [X_i \mid Y_A(t), Y_B(t)] - E [X_i \mid Y_A(t), Y'_B(t)])^2 \right],$$

where $A = [N] \setminus \{i\} \cup B$ and $Y(t)$ denotes an independent second use of the BMS channel with the same input $X$.

Our approach to bounding this term is to view the input vector $(X_0, \ldots, X_{N-1})$, as the first $N$ entries in an extended input vector $(X_0, \ldots, X_{M-1})$ of length $M > N$. With some abuse of notation we use $X_{[N]}$ to denote the original input vector and $X_{[M]}$ to denote the extended input vector. Associated with the extended input we define the output $Y_{[M]}(t) = (Y_0(t), \ldots, Y_{M-1}(t))$ from the same BMS channel. If we can find an extension such that:

i) the extended input $X_{[M]}$ is distributed uniformly on the codewords of a linear code; and

ii) there exists a set $C \subseteq \{N, \ldots, M-1\}$ and permutation matrix $\Pi$ such that $(X_i, X_A, X_B)$ is equal in distribution to $(X_i, X_A, \Pi X_C)$,

then we can use Lemma 25 to conclude that

$$\Delta^B(t) = \frac{1}{2} E \left[ (E [X_i \mid Y_A(t), Y_B(t)] - E [X_i \mid Y_A(t), Y'_B(t)])^2 \right].$$

(22)

In words, the second look at the entries indexed by $B$ has been replaced by observations of the entries in the extended codeword indexed by $C$.

To apply this, we assume that the original input is generated by a uniform distribution over the codewords of $RM(r, m)$ and the extended input is generated by a uniform distribution over the codewords of $RM(r, m+k)$, for some positive integer $k \leq m$. Then, the following lemma shows that the nesting property identified in Lemma 8 can be used to choose the sets $A$, $B$, and $C$ to satisfy the distributional condition defined in Lemma 25. See Figure 2 for an illustration.

**Lemma 28.** For positive integers $(r, m, k)$ with $r, k \leq m$, let $N = 2^m$ and $M = 2^{m+k}$. If $X_{[M]}$ is distributed uniformly on the codewords of the $RM(r, m+k)$ code then $X_{[N]}$ is distributed uniformly on the codewords of the $RM(r, m)$ code. Furthermore, for each $i \in [N]$ there exists a partition $\{i\}, A, B$ of $\emptyset, BMS \text{ channel}$, and a permutation matrix $\Pi$ such that $(X_i, X_A, X_B)$ is equal in distribution to $(X_i, X_A, \Pi X_C)$.

**Proof.** If $X_{[M]}$ is uniformly distributed on the codewords of $C = RM(r, m+k)$ and $I = [N]$, then Lemma 8 shows that $X_I = X_{[N]}$ is uniformly distributed on the codewords of $C_I = RM(r, m)$. For $i = 0$, the sets $A, B, C$ are also constructed in Lemma 8 and we will verify their properties below. At the end, we describe how the $i = 0$ construction can be remapped to any $i \in [N]$.

To explain why the sets $A, B, C$ from Lemma 8 satisfy the stated conditions, we recall their definition from the proof of Lemma 8. First, we define $V = F_2^m \times \{0\}^k$ and $V' = F_2^{m-k} \times \{0\}^k \times F_2^k$ as the evaluation sets associated with the indices $I = \tau(V)$ and $I' = \tau(V')$, respectively. Then, we define $T = I \cap I' = [2^m-k], A = T \setminus \{0\}, B = I \setminus T$, and $C = I' \setminus T$. To simplify notation, we also define $A' = \tau^{-1}(A), B' = \tau^{-1}(B)$, and $C' = \tau^{-1}(C)$.

Consider the function $\pi : F_2^{m+k} \to F_2^{m+k}$ defined by $v \mapsto v'$ with $v'_i = v_i$ for $i \in [m-k], v'_i = v_{i+k}$ for $i \in \{m-k, \ldots, m-1\}$, and $v'_i = v_{i-k}$ for $i \in \{m, \ldots, m+k-1\}$. This function simply swaps bits $v_{m-k+i}$ and $v_{m+i}$ for all $i \in [k]$. One can verify that $\pi$ is a permutation on $F_2^{m+k}$ that satisfies $\pi(0) = 0, \pi(\pi(0)) = 0$ for all $v \in F_2^{m+k}, \pi(v) = v$ for all $v \in A'$, and $\pi(B') = C'$. We do not discuss the precise element by element mapping from $B'$ to $C'$ but that is fine because Lemma 25 allows for an arbitrary permutation of the set $C$.

Since $\pi$ is a linear function, it defines an automorphism of $C$ [p. 398]. For integer indices, this automorphism is given by $i \mapsto \pi(\pi^{-1}(i))$. Also, it is easy to verify that a code automorphism naturally preserves a uniform distribution over the codewords. Thus, the above statements imply that $(X_0, X_A, X_B)$ is equal in distribution to $(X_0, X_A, \Pi X_C)$ for some permutation matrix $\Pi$. 

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For $i \in [2^n] \setminus \{0\}$, we can simply translate the sets $A', B', \text{ and } C'$ by adding $i' = \tau^{-1}(i)$. In particular, we define $\pi_i(v) = \pi(v-i') + i'$, $A'_i = A' + i'$, $B'_i = B' + i'$, and $C'_i = C' + i'$. Observe also that $\{i'\}, A'_i, B'_i$ forms a partition of $V$. Next, one can verify that $\pi_i$ is a permutation on $\mathbb{F}_2^{m+k}$ that satisfies $\pi_i(i') = i'$, $\pi_i(v) = v$ for all $v \in \mathbb{F}_2^{m+k}$, and $\pi_i(B'_i) = C'_i$. Like before, since $\pi_i$ is an affine function on $\mathbb{F}_2^{m+k}$, it defines an automorphism of $C$ that preserves the uniform distribution over codewords. For $i \in [N]$, we define $A_i = \tau(A'_i)$, $B_i = \tau(B'_i)$, and $C_i = \tau(C'_i)$. Then, the above statements imply that $\{i\}, A_i, B_i$ forms a partition of $[N]$ and that $(X_i, X_{A_i}, X_{B_i})$ is equal in distribution to $(X_i, X_{A_i}, \Pi_i X_{C_i})$ for some permutation matrix $\Pi_i$. 

Starting with (22), an application of the estimation inequality in Lemma 12 followed by the data processing inequality for MMSE allows us to write

$$
\Delta_i^B(t) \leq \text{mmse}(X_i | Y_A(t), Y_B(t)) + \text{mmse}(X_i | Y_A(t), Y_C(t)) - 2 \text{mmse}(X_i | Y_B(t), Y_C(t))
$$

$$
= 2(\text{mmse}(X_i | Y_{[N] \setminus \{i\}}(t)) - \text{mmse}(X_i | Y_A(t), Y_B(t), Y_C(t)))
$$

$$
\leq 2(\text{mmse}(X_i | Y_{[N] \setminus \{i\}}(t)) - \text{mmse}(X_i | Y_{[M] \setminus \{i\}}(t)))
$$

Now, we can integrate to get

$$
\int_0^1 \Delta_i^B(t) \, dt \leq \int_0^1 2 \left( \text{mmse}(X_i | Y_{[N] \setminus \{i\}}(t)) - \text{mmse}(X_i | Y_{[M] \setminus \{i\}}(t)) \right) \, dt
$$

$$
\leq 4 \ln 2 \int_0^1 \frac{d}{dt} \left( \frac{H(X_{[N]} | Y_{[N]}(t))}{N} - \frac{H(X_{[M]} | Y_{[M]}(t))}{M} \right) \, dt
$$

$$
= \frac{4 \ln 2}{N} \left( \frac{H(X_{[N]})}{N} - \frac{H(X_{[M]})}{M} \right)
$$

$$
= \frac{4 \ln 2}{M} \left( R(r, m) - R(r, m + k) \right),
$$

where the second inequality follows from applying Lemma 20 and observing that $X_{[N]}$ and $X_{[M]}$ both have transitive symmetry groups. This concludes the proof of (20).

### 5.2.3 Single-Term Bound

This section proves an upper bound on the integral of the term $\Delta_i^j(t)$ defined in (18). Recall that this term is given by

$$
\Delta_i^j(t) := \frac{1}{2} \mathbb{E} \left[ \left( \mathbb{E}[X_i | Y_{\omega_i}(t)] - \mathbb{E}[X_i | Y_{\omega_i}^j(t)] \right)^2 \right],
$$

where $Y^j(t)$ is a version of $Y(t)$ with the $j$-th output resampled. Using the estimation inequality in Lemma 12 we can bound this term according to

$$
\Delta_i^j(t) \leq \text{mmse}(X_i | Y_{\omega_i}(t)) + \text{mmse}(X_i | Y_{\omega_i}^j(t)) - 2 \text{mmse}(X_i | Y_{\omega_i}(t), Y_{\omega_i}^j(t))
$$

$$
= 2 \left( \text{mmse}(X_i | Y_{\omega_i}(t)) - \text{mmse}(X_i | Y_{\omega_i}(t), Y_{\omega_i}^j(t)) \right)
$$

(23)

where $Y_{\omega_i}^j(t)$ is an independent second use of the $j$-th channel. Here, we recognize that the first term is simply the extrinsic MMSE function $M_i(t)$. The second term can be interpreted as the extrinsic MMSE of an augmented channel that gets an independent second observation of the $j$-th channel. To bound the difference of these MMSE terms, we introduce a modified version of the GEXIT function corresponding to this augmented channel.

**Definition 29** (Augmented GEXIT). For $i, j \in [N]$ we define $G_{ij}^j : [0, 1] \to \mathbb{R}$ to be the GEXIT function of an augmented family of BMS channel that uses the $j$-th channel twice. Specifically, letting $Y_{ij}^j(t)$ denote an independent second observation of the $j$-th input, we have

$$
\begin{align*}
G_{ij}^j(t) &= \left. \frac{\partial}{\partial s} H(X_i | Y_i(s), Y_{\omega_i}(t), Y_{ij}^j(t)) \right|_{s=t}, \quad i \neq j \\
&= \left. \frac{\partial}{\partial s} H(X_i | Y_i(s), Y_{\omega_i}(t), Y_{ij}^j(s)) \right|_{s=t}, \quad i = j.
\end{align*}
$$
By Lemma 20 and the assumptions on the BMS channel, the partial derivatives in the definition of $G_i(t)$ and $G'_i(t)$ exist almost everywhere on $[0, 1]$. For $i \neq j$, we can apply the I-MMSE relation given in (12) to write

$$G_i(t) - G'_i(t) \geq \frac{1}{2 \ln 2} \left( \text{mmse}(X_i \mid Y_{-i}) - \text{mmse}(X_i \mid Y_{-i}, Y'_j) \right),$$

for almost all $t \in [0, 1]$, where we recall that $\mathcal{M}'(t) = 1$ under the assumed parametrization of the channel family.

In view of (23) and (24), we see that the integral of the difference in GEXIT functions provides an upper bound on the integral of $\Delta_j^i(t)$. If the input distribution has a transitive symmetry group, then the integral of $G_i(t)$ follows directly from the definition of the GEXIT function as discussed in Section 3.3. However, the integral of $G'_i(t)$ does not have such a simple interpretation because the partial derivative of the augmented channel with respect to $j$ is different than for the other channels.

The next lemma provides a bound on the integral in question, averaged over the indices $i \neq j$. If the input distribution has a doubly transitive symmetry, then this gives a bound that holds uniformly for all pairs of indices. Combining this result with the bounds in (23) and (24) gives the single term bound stated in (21).

**Lemma 30.** For every input distribution on $\{\pm 1\}^N$,

$$\sum_{i,j \in [N]: i \neq j} \int_0^1 (G_i(t) - G'_i(t)) \, dt \leq N.$$  

In particular, if the input distribution has doubly transitive symmetry then $G_i^j = G_k^\ell$ for all $i, j, k, \ell \in [N]$ with $i \neq j$ and $k \neq \ell$ and so

$$\int_0^1 (G_i(t) - G'_i(t)) \, dt \leq \frac{1}{N - 1}, \quad i, j \in [N], \quad i \neq j.$$  

**Proof.** See Section 6.3.

### 5.3 Bounds on the Extrinsic MMSE via the Area Theorem

Having established the sharp threshold phenomenon in the sense of (16), the next step is to provide bounds on the extrinsic MMSE in terms of the rate of the code. The key tool that enables this is a relation known as the area theorem for GEXIT functions [31, 32]. Consider a family of BMS channels satisfying the assumptions in Definition 15 and let $G(t)$ be the GEXIT function associated with a random input $X$ of length $N$ whose distribution has a transitive symmetry group. Then, the generalized area theorem [4] implies that

$$\frac{1}{N} H(X) = \int_0^1 G(t) \, dt.$$  

This statement is an immediate consequence of the definition of the GEXIT function and the assumption of a transitive symmetry, which ensures that GEXIT function is the same for all inputs. If the distribution of $X$ is uniformly distributed over the input sequences of a binary code, then the LHS of (27) is the rate of the code.

For the purposes of this paper, the connection between the rate and the extrinsic MMSE follows from the results in Section 4.2. The details are summarized in the following result, which provides bounds on the MMSE in terms of the integral appearing in (16) and the gap between the Shannon capacity and the code rate.

**Lemma 31.** Consider a family of BMS channels satisfying Assumption 1 and suppose that the input distribution is uniform over a code with transitive symmetry and rate $R$. There exists a unique value $t_R \in (0, 1)$ such that $C(t_R) = R$. Furthermore, the extrinsic MMSE satisfies

$$M(t) \leq \frac{\kappa(t)}{C(t) - R} \int_0^t M(s)(1 - M(s)) \, ds, \quad t \in [0, t_R).$$  

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where \( \kappa(t):=\sup_{s\in[0,1]} \mathcal{H}'(s), \) \( \psi(u):=1-(1-2h_{b}^{-1}(u))^{2}, \) and \( h_{b}^{-1}:[0,1]\to[0,1/2] \) is the inverse of the binary entropy function restricted to the domain \([0,1/2]. \) The function \( \psi \) is non-negative and strictly increasing. Thus, the denominator in (29) is strictly positive for \( t\in(t_{R},1]. \)

**Proof.** See Section 6.3

If the input is defined by an RM code, then we can combine Lemma 31 with Lemma 27 to obtain bounds on the extrinsic MMSE that depend only on the code rate and the blocklength. Applying these bounds to a sequence of RM codes with strictly increasing blocklength and code rate converging to \( R\in(0,1), \) shows that the extrinsic MMSE converges to a 0-1 step function that jumps at the unique \( t_{R}\in(0,1) \) such that \( C(t_{R})=R. \)

**Remark 32.** In Appendix A.4, we use an alternative approach to establish the limiting behavior of the extrinsic MMSE for a sequence of RM codes. In particular, by using the comparisons in Lemma 33, one can avoid the need for explicit bounds. While the proofs are not necessarily shorter or simpler, we believe that the approach may be of independent interest.

**Example 33.** To help explain the upper bound in Lemma 31 we describe an extrinsic MMSE curve that satisfies the bound with equality. This shows that (28) is tight in the sense that it cannot be improved without imposing some additional constraints on the extrinsic MMSE function. Consider the family of BECs with erasure probability equal to \( t \) and observe that \( \mathcal{H}(t)=t \) and \( \kappa(t)=1. \) Let \( M(t) \) be the extrinsic MMSE function for a code with rate \( R \) and a transitive symmetry group, and define \( \rho:=\int_{0}^{t} M(s)(1-M(s))\,ds. \) Then, for each \( t^{*}\in[0,1-R-\rho) \) there exists a non-decreasing function \( \tilde{M}: [0,1]\to[0,1] \) with \( \tilde{M}(0)=0 \) and \( \tilde{M}(1)=1 \) with the following properties:

- The integral constraint implied by \( \rho \) holds with equality, i.e., \( \int_{0}^{t^{*}} \tilde{M}(s)(1-\tilde{M}(s))\,ds=\rho. \)

- The inequality implied by the area theorem (27) and Lemma 21 namely \( \int_{0}^{t} \tilde{M}(s)\mathcal{H}'(s)\,ds\leq R, \) holds with equality because \( \int_{0}^{t} \tilde{M}(s)\,ds=R \) and \( \mathcal{H}'(s)=1 \) for the family of BECs.

![Illustration of the upper bound (red) on extrinsic MMSE given by (28) for a rate \( R=0.5 \) code on a BEC channel with erasure rate \( t \) when \( \rho:=\int_{0}^{1} M(s)(1-M(s))\,ds=0.02. \) This bound is sharp because, for any \( t^{*}\in[0,1-R-\rho), \) there is a non-decreasing function \( \tilde{M}: [0,1]\to[0,1] \) that equals the upper bound at \( t=t^{*} \) and also satisfies the area theorem (e.g., the area in blue is equal to \( R) \) and the integral constraint \( \int_{0}^{t} M(s)(1-M(s))\,ds=\rho. \) The function \( \tilde{M}(t) \) is shown (blue) for \( t^{*}=0.4 \) and is given by (30) with \( u^{*}=0.525. \) See Example 33 for more details.](image-url)
• The upper bound in \(28\) is attained at the point \(t^*\), i.e., \(\tilde{M}(t^*) = \rho/(C(t^*) - R)\) where we recall that \(\kappa(t) = 1\) for the family of BECs.

Specifically, this function is given by

\[
\tilde{M}(t) = \begin{cases} 
0 & t \in [0, t^*) \\
\frac{\rho}{C(t^*) - R} & t \in [t^*, u^*) \\
1, & t \in [u^*, 1] 
\end{cases} 
\]  
(30)

where \(u^* = 1 - R + \rho(C(t^*) - R)/(C(t^*) - R - \rho)\). An example is shown in Figure 4.

5.4 RM Codes Achieve Capacity on BMS Channels

We are now ready to prove the main result of the paper. To show that RM codes achieve capacity for any particular BMS channel, we need to show that the channel can be embedded into a family of BMS channels satisfying Assumption 1. To this end, we may consider the following construction.

**Definition 34** (Interpolated family of BMS channels). For a BMS channel with input alphabet \(\mathcal{X} = \{\pm 1\}\), output alphabet \(\mathcal{Y}\), and capacity \(C \in (0, 1)\) an interpolated family of BMS channels satisfying Assumption 1 is defined by the following steps:

- Let \(\mathcal{M}^* \in (0, 1)\) be the MMSE of the channel associated with a uniform input distribution.
- For \(0 \leq t < \mathcal{M}^*\) the output is given by the original channel with probability \(t/\mathcal{M}^*\) and perfect knowledge of the input otherwise. This can be accomplished, for example, by adding the symbols \(\pm \infty\) to \(\mathcal{Y}\) and associating them with inputs \(\pm 1\), respectively.
- For \(\mathcal{M}^* \leq t \leq 1\) the output is given by the original channel with probability \((1 - t)/(1 - \mathcal{M}^*)\) and is equal to the erasure symbol otherwise.

The MMSE function is \(\mathcal{M}(t) = t\) and the entropy function is

\[
\mathcal{H}(t) = \begin{cases} 
\frac{t}{\mathcal{M}^*}(1 - C), & t \in [0, \mathcal{M}^*) \\
\frac{t - \mathcal{M}^*}{1 - \mathcal{M}^*}C + 1 - C, & t \in [\mathcal{M}^*, 1] 
\end{cases} 
\]

The original BMS channel corresponds to the point \(t = \mathcal{M}^*\).

The next result bounds the extrinsic MMSE for RM codes transmitted over a BMS channel.

**Lemma 35.** Consider a BMS channel with capacity \(C \in (0, 1)\). The extrinsic MMSE of an RM\((r, m)\) code with rate \(R = R(r, m)\) satisfies

\[
\begin{align*}
\text{mmse}(X_i | Y_{-i}) &\leq \frac{\rho(m)}{C - R}, & R < C \\
\text{mmse}(X_i | Y_{-i}) &\geq 1 - \frac{(1 - C)\rho(m)}{\psi(1 - C)\Psi(R - C)}, & R > C
\end{align*}
\]

for all \(i \in [N]\) where \(\rho(m) := (6\ln(m) + 34)/(5\sqrt{m})\) and \(\Psi(u) := \int_0^u \psi(v)dv\) with \(\psi\) given in Lemma 31.

**Proof.** Let \(\mathcal{M}^*\) be MMSE of the given BMS channel when the input is uniformly distributed on \(\{\pm 1\}\). Let \(\mathcal{H}(t)\) be the entropy function associated with the family of channels in Definition 34, and let \(t_R\) be the unique value such that \(1 - \mathcal{H}(t_R) = R\).

For the upper bound, observe that if \(R < C\) then \(t_R > \mathcal{M}^*\). Combining the bound in (28), evaluated at \(t = \mathcal{M}^*\), with the bound on \(\int_0^1 M(s)(1 - M(s))ds\) in Lemma 27 we see that

\[
\text{mmse}(X_i | Y_{-i}) \leq \frac{\kappa(\mathcal{M}^*)\rho(m)}{C - R} \leq \frac{C\rho(m)}{(1 - \mathcal{M}^*)(C - R)}
\]
where the second step holds because \( \mathcal{H}'(s) = C/(1 - \mathcal{M}^*) \) for \( s \geq \mathcal{M}^* \).

For the lower bound, observe that, if \( R > C \), then \( t_R < \mathcal{M}^* \). Combining the bound in (29), evaluated at \( t = \mathcal{M}^* \), with Lemma 27, we see that

\[
\text{mmse}(X_i \mid Y_{-i}) \geq 1 - \frac{(1 - C)\rho(m)}{\mathcal{M}^*\Psi(R - C)}
\]

where the second step follows from a change of variables in the integral.

Finally, we simplify the bound to avoid dependence on \( \mathcal{M}^* \). Notice that (7) in Section 4.1 implies

\[
\mathcal{M}^* \leq 1 - C \leq h_B \left( \frac{1 - \sqrt{1 - \mathcal{M}^*}}{2} \right),
\]

where the inequality on the right is equivalent to \( \psi(1 - C) \leq \mathcal{M}^* \).

We now state the main result of the paper, which provides non-asymptotic bounds on the BER under bit-MAP decoding for an RM code over a BMS channel. These bounds depend only on three quantities: the capacity of the channel, the difference between the capacity and the code rate, and the blocklength. Evaluating these bounds in the limit of increasing blocklength, it follows that RM codes achieve capacity on any BMS channel.

**Theorem 36.** Consider a BMS channel with capacity \( C \in (0, 1) \). For every \( \text{RM}(r, m) \) code whose rate satisfies \( R(r, m) < C \), the bit-error rate under bit-MAP decoding satisfies

\[
\text{BER}(X_i \mid Y) \leq \frac{\frac{1}{2}\rho(m)}{C(t) - R(r, m)},
\]

for all \( i \in [N] \) where \( \rho(m) := (6 \ln(m) + 34)/(5\sqrt{m}) \). In particular, for every \( R \in [0, C) \) there exists a sequence of RM codes with increasing blocklength and rate converging to \( R \) such that the BER under bit-MAP decoding converges to zero.

Conversely, if \( R(r, m) > C \) then

\[
\text{BER}(X_i \mid Y) \geq \text{BER}(X_i \mid Y_i) - \sqrt{\frac{\ln(2)(1 - C)\rho(m)}{2\psi(1 - C)\Psi(R(r, m) - C)}},
\]

for all \( i \in [N] \) where \( \Psi(u) := \int_0^u \psi(v) \, dv \) with \( \psi \) given in Lemma 31. In particular, for every \( R \in (C, 1] \) and every sequence of RM codes with increasing blocklength and rate converging to \( R \), the BER under bit-MAP decoding converges to the bit-error rate associated with a single use of the channel.

**Proof.** The upper bound on the BER follows from combining the upper bound on the extrinsic MMSE in Lemma 35 with the relationship between the BER and MMSE in Lemma 11 and then noting that \( \text{BER}(X_i \mid Y) \leq \text{BER}(X_i \mid Y_{-i}) \). The lower bound on the BER follows from combining the lower bound on the extrinsic MMSE in Lemma 35 with the relationship between the BER and MMSE in Lemma 19.

Finally, from [7, Remark 24], we know that for any \( R \in (0, 1) \), there is a sequence of RM codes with strictly increasing \( m \) whose rate converges to \( R \). The construction of this sequence is also discussed in Section 2.1 for completeness. \( \square \)

6 Proofs

In this section, we collect proofs that have been removed from the main text due to length or importance.
6.1 Background

Proof of Lemma 11. We start by observing that

\[
\text{BER}(X \mid Y) := 1 - \mathbb{E} \left[ \max \{ \Pr(X = 1 \mid Y), \Pr(X = -1 \mid Y) \} \right]
\]

\[
= 1 - \mathbb{E} \left[ \frac{\Pr(X = 1 \mid Y) + \Pr(X = -1 \mid Y)}{2} + \frac{\Pr(X = 1 \mid Y) - \Pr(X = -1 \mid Y)}{2} \right]
\]

\[
= \frac{1}{2} \left( 1 - \mathbb{E} \left[ |E[X \mid Y]| \right] \right),
\]

where the first step follows from \( \max \{a, b\} = \frac{1}{2}(a + b + |a - b|) \). Similarly, MMSE can be written as

\[
\text{mmse}(X \mid Y) = 1 - \mathbb{E} \left[ (XE[X \mid Y])^2 \right]
\]

\[
= 1 - \mathbb{E} \left[ |E[X \mid Y]|^2 \right].
\]

Thus, the inequality \( 1 - u \leq 1 - u^2 \) for \( u \in [0, 1] \) implies that

\[
\text{BER}(X \mid Y) \leq \frac{1}{2} \text{mmse}(X \mid Y),
\]

with equality if and only if \( \mathbb{E}[X \mid Y] \in \{0, \pm 1\} \) (i.e., the channel is equivalent to an erasure channel). Alternatively, by Jensen’s inequality

\[
1 - 2 \text{BER}(X \mid Y) \leq \sqrt{1 - \text{mmse}(X \mid Y)}
\]

and so

\[
\text{BER}(X \mid Y) \geq \frac{1 - \sqrt{1 - \text{mmse}(X \mid Y)}}{2},
\]

with equality if and only if \( \mathbb{E}[X \mid Y] \) has constant magnitude (i.e., the channel is equivalent to a BSC). Thus, for a sequence of observations, the bit-error probability approaches 0 (respectively \( \frac{1}{2} \)) if and only if the MMSE approaches 0 (respectively 1). \( \square \)

Proof of Lemma 12. We start by writing

\[
4 \text{mmse}(X \mid Y, Y')
\]

\[
\leq 4 \mathbb{E} \left[ \left( X - \mathbb{E}[X \mid Y] + \mathbb{E}[X \mid Y'] \right)^2 \right]
\]

\[
= \mathbb{E} \left[ (X - \mathbb{E}[X \mid Y] + X - \mathbb{E}[X \mid Y'])^2 \right]
\]

\[
= \text{mmse}(X \mid Y) + \text{mmse}(X \mid Y') + 2 \mathbb{E} [ (X - \mathbb{E}[X \mid Y]) (X - \mathbb{E}[X \mid Y'])],
\]

where the first inequality holds because averaging the individual estimates may be suboptimal. One can bound the expectation on the last line by using Cauchy-Schwarz to see that

\[
|\mathbb{E} [(X - \mathbb{E}[X \mid Y]) (X - \mathbb{E}[X \mid Y'])]| \leq \sqrt{\text{mmse}(X \mid Y) \text{mmse}(X \mid Y')}. \]

Next, we write that

\[
\mathbb{E} \left[ (\mathbb{E}[X \mid Y']) - \mathbb{E}[X \mid Y]\right]^2
\]

\[
= \mathbb{E} \left[ ((X - \mathbb{E}[X \mid Y]) - (X - \mathbb{E}[X \mid Y']))^2 \right]
\]

\[
= \text{mmse}(X \mid Y) + \text{mmse}(X \mid Y') - 2 \mathbb{E} [(X - \mathbb{E}[X \mid Y]) (X - \mathbb{E}[X \mid Y'])].
\]

Putting these together, we see that

\[
\mathbb{E} \left[ (\mathbb{E}[X \mid Y'] - \mathbb{E}[X \mid Y])^2 \right]
\]

\[
= \text{mmse}(X \mid Y) + \text{mmse}(X \mid Y') - 2 \mathbb{E} [(X - \mathbb{E}[X \mid Y]) (X - \mathbb{E}[X \mid Y'])]
\]

\[
\leq \text{mmse}(X \mid Y) + \text{mmse}(X \mid Y') - 4 \text{mmse}(X \mid Y, Y')
\]

\[
= 2 \text{mmse}(X \mid Y) + 2 \text{mmse}(X \mid Y') - 4 \text{mmse}(X \mid Y, Y'). \]
6.2 Preliminary Results

Proof of Lemma \[16\]. First, we recall that \( h_b : [0, 1] \rightarrow [0, 1] \) with \( h_b(x) := x \log_2 \frac{1}{x} + (1 - x) \log_2 \frac{1}{1-x} \) defines the binary entropy function with the convention that \( h_b(0) = h_b(1) = 0 \). Then, we observe that

\[
H(X|Y) = E \left[ h_b \left( \Pr(X = -1|Y) \right) \right] = E \left[ h_b \left( \frac{1 - E[Y|X]}{2} \right) \right]
\]

because \( E[X|Y] = 1 - 2 \Pr(X = -1|Y) \). Next, we compute the series expansion of \( h_b \left( \frac{1-m}{2} \right) \) about \( m = 0 \) with

\[
\begin{align*}
    h_b \left( \frac{1-m}{2} \right) &= \frac{1}{\ln 2} \left( -\frac{1-m}{2} \ln \frac{1-m}{2} - \frac{1+m}{2} \ln \frac{1+m}{2} \right) \\
    &= 1 - \frac{1}{2\ln 2} \left( (1-m) \ln(1-m) + (1+m) \ln(1+m) \right) \\
    &= 1 - \frac{1}{2\ln 2} \left( -(1-m) \sum_{j=1}^{\infty} \frac{m^j}{j} - (1+m) \sum_{j=1}^{\infty} \frac{(-m)^j}{j} \right) \\
    &= 1 - \frac{1}{2\ln 2} \sum_{j=1}^{\infty} \left( \frac{m^{j+1} + (-m)^{j+1}}{j} - \frac{m^j + (-m)^j}{j} \right) \\
    &= 1 - \frac{1}{2\ln 2} \sum_{k=1}^{\infty} \left( \frac{2m^{2k}}{2k-1} - \frac{2m^{2k}}{2k} \right) \\
    &= 1 - \frac{1}{\ln 2} \sum_{k=1}^{\infty} \frac{m^{2k}}{2k(2k-1)},
\end{align*}
\]

where each step holds for all \(|m| < 1\) and the equality of the first and last expression can be verified separately for \( m \in \{0, 1\} \). Since the last infinite sum converges uniformly to \( h_b \left( \frac{1-m}{2} \right) \) for \(|m| \leq 1\) and \(|E[X|Y]| \leq 1\), we can take the expectation of both sides. Finally, we obtain the stated result by observing that \( \sum_{k=1}^{\infty} (2k(2k-1))^{-1} = \ln 2 \).

For the second result, we define \( Z = X \cdot X' \) and use the power series representation to verify the formula because

\[
H(Z|Y,Y') = E \left[ h_b \left( \frac{1 - E[Z|Y,Y']}{2} \right) \right]
\]

and the conditional independence of the channels implies that

\[
E[Z|Y,Y'] = E[X \cdot X'|Y,Y'] = E[X|Y]E[X'|Y'].
\]

\hfill\Box

Proof of Lemma \[17\]. Without loss of generality we may assume that the channel \( W_{Y|X} \) has a real-valued output satisfying the conditions given in Definition \[10\] i.e., the transition probability satisfies \( w(y \mid +1) = w(-y \mid -1) \). Let \( V \in \{ \pm 1 \} \) be a uniform random variable that is independent of \((X,Y,Z)\) and define \( X^* = VX \) and \( Y^* = VY \). By construction, the distribution of \((X^*,Y^*)\) is equal to the distribution of an input-output pair from \( W_{Y|X} \) where the input \( X^* \) is uniformly distributed on \( \{ \pm 1 \} \). This is because the output \( Y^* = X^*(XY) \) has the same multiplicative noise structure as \( W_{Y|X} \) but the noise is applied to the input \( X^* \). Moreover, the pairs \((X,Z)\) and \((X^*,Y^*)\) are independent because \( X^* \), \( X \), and \( XY \) are independent and \( Y - X - Z \) forms a Markov chain.

Using the chain rule for entropy, we can now write

\[
H(X \mid Z, Y) = H(X \mid Z, Y, V) \\
= H(X \mid Z, Y^*, V) \\
= H(X, V \mid Z, Y^*) - H(V \mid Z, Y^*) \\
= H(X, X^* \mid Z, Y^*) - H(V \mid Z, Y^*) \\
= H(X \mid Z) + H(X^* \mid Y^*) - H(XX^* \mid Z, Y^*).
\]
For the bound on the MMSE, we use the entropy expansion in Lemma 16 twice to write

the lower bound on the conditional mutual information:

Taking the difference and then noting that all the terms in the summation are non-negative leads to a

proof of Lemma 19. Starting with the two-look formula (9) and noting that \( E[X \mid Z]^{2k} \leq E[X \mid Z]^2 \) for all \( k \geq 1 \), we see that

for the BER, we use the identity \( \text{BER}(X \mid Z) = \frac{1}{2} \left( 1 - E[|E[X \mid Z]|] \right) \), which is derived in the

For the next few results, the following definition and lemma will be useful.

Here, the fourth step follows because entropy is invariant under the one-to-one transformation \((X, V) \mapsto (X, X^*)\) with \( X^* = XV \). The last step follows from the independence of \((X, Z)\) and \((X^*, Y^*)\). Applying the entropy decompositions given in Lemma 16 to the terms on the RHS and then simplifying the terms under a single summation gives the desired result.

The desired result follows from combining this inequality with

where the second step follows from the reverse triangle inequality and the third step uses Jensen’s inequality. In view of the bound on the MMSE, the proof is complete.

For the next few results, the following definition and lemma will be useful.
Definition 37 (Absolutely Continuous). Consider a real interval \([a, b]\) and a function \(f: [a, b] \rightarrow \mathbb{R}\). Then, \(f\) is absolutely continuous on \([a, b]\) if, for every \(\epsilon > 0\), there is a \(\delta > 0\) such that, for any sequence of disjoint intervals \(\{[a_k, b_k]\}_{k \in \mathbb{N}}\) with \(a \leq a_k \leq b_k \leq b\), we have

\[
\sum_{k \in \mathbb{N}} |b_k - a_k| < \delta \implies \sum_{k \in \mathbb{N}} |f(b_k) - f(a_k)| < \epsilon.
\]

This definition is important because the fundamental theorem of calculus for the Lebesgue integral states that, if \(f\) is absolutely continuous, then \(f\) is differentiable almost everywhere on \([a, b]\) and, for all \(c \in [a, b]\), the Lebesgue integral of its derivative satisfies

\[
f(c) = f(a) + \int_a^c f'(x)dx.
\]

Lemma 38. Consider a function \(f: [a, b]\) that is absolutely continuous on \([a, b]\) and another function \(g: [a, b] \rightarrow \mathbb{R}\). Then, if there is a constant \(\gamma < \infty\) such that \(|g(y) - g(x)| \leq \gamma|f(y) - f(x)|\) for all \(x, y \in [a, b]\), then \(g\) is absolutely continuous on \([a, b]\).

Proof. For any \(\epsilon' > 0\), we use the absolute continuity of \(f\) with \(\epsilon = \epsilon'/\gamma\) to obtain the desired \(\delta > 0\). Thus, we find that, for any sequence of disjoint intervals \(\{[a_k, b_k]\}_{k \in \mathbb{N}}\) with \(a \leq a_k \leq b_k \leq b\), we have

\[
\sum_{k \in \mathbb{N}} |b_k - a_k| < \delta \implies \sum_{k \in \mathbb{N}} |g(b_k) - g(a_k)| \leq \gamma \sum_{k \in \mathbb{N}} |f(b_k) - f(a_k)| < \epsilon'.
\]

Proof of Lemma 38. Without loss of generality we may assume that the channel \(W(t)\) has a real-valued output satisfying the conditions given in Definition 10, i.e., the transition probability satisfies \(w(y \mid +1) = w(-y \mid -1)\). We will also denote \(Y(t)\) by \(Y\) unless the parameter \(t\) is important the discussion.

Let \(V \in \{\pm 1\}\) be a uniform random variable that is independent of \((X, Y, Z, Z_1, Z_2)\) and define \(X^* = V X\) and \(Y^* = V Y\). This setup mirrors the proof of Lemma 17 and its properties are described there. Using this, we can apply Lemma 17 to see that

\[
H(X|Z, Y) = \frac{1}{\ln 2} \sum_{k=1}^{\infty} \frac{1}{2^{k(2k-1)}} \left[1 - E[|X|^2] - E[E[X^*Y^*]|2^k] + E[E[X[Y^*]|2^k] E[E[X^*Y^*]|2^k]\right].
\]

Since \(E[X|Z]\) does not depend on \(t\), it follows that, for \(0 \leq t \leq t' \leq 1\), we have

\[
H(X|Z, Y(t')) - H(X|Z, Y(t)) = \frac{1}{\ln 2} \sum_{k=1}^{\infty} \frac{1}{2^{k(2k-1)}} \left(E[E[X^*Y^*]|2^k] - E[E[X^*Y^*]|2^k]\right)
\]

by Lemma 38

\[
\leq \frac{1}{\ln 2} \sum_{k=1}^{\infty} \frac{1}{2^{k(2k-1)}} \left(E[E[X^*Y^*]|2^k] - E[E[X^*Y^*]|2^k]\right) = \mathcal{H}(t') - \mathcal{H}(t),
\]

where the inequality follows from the moment bound \(0 \leq E[|X|^2] \leq 1\) and the ordering of moments due to degradation. Since \(H(X|Z, Y(t))\) is increasing, the LHS equals its absolute value and hence

\[
|H(X|Z, Y(t')) - H(X|Z, Y(t))| \leq |\mathcal{H}(t') - \mathcal{H}(t)|.
\]

Since \(W(t)\) is absolutely continuous according to Definition 15, it follows that \(\mathcal{H}(t)\) is absolutely continuous and we can apply Lemma 38 to see that \(H(X|Z, Y(t))\) is also absolutely continuous. Thus, \(\frac{d}{dt} H(X|Z, Y(t))\) exists for almost all \(t \in [0, 1]\) and satisfies the fundamental theorem of calculus.

If the above moment bound is only applied to the \(k \geq 2\) terms and the \(k = 1\) term is unchanged, then the upper bound changes to

\[
H(X|Z, Y(t')) - H(X|Z, Y(t))
\]
Since $\mathcal{M}(t)$ is also absolutely continuous (e.g., see \cite{8}), the three functions in the above expression are absolutely continuous and their derivatives exist simultaneously for almost all $t \in [0, 1]$. Thus, for such a $t$, we can choose $t' = t + \delta$, divide by $\delta$, and take the limit at $\delta \to 0$ to see that

$$
\frac{d}{dt} H(X|Z, Y(t)) \leq \frac{(\text{mmse}(X|Z) - 1)(\mathcal{M}(t') - \mathcal{M}(t))}{2\ln 2} + \mathcal{H}(t') - \mathcal{H}(t).
$$

This bound is useful when $\text{mmse}(X|Z)$ is close to 1. When $\text{mmse}(X|Z)$ is close to 0, one can take a different approach. In particular, combining \cite{31} and $\mathbb{E}[E[X|Z]^{2k}] \geq \mathbb{E}[E[X|Z]^2]^k$ shows that

$$
H(X|Z, Y(t')) - H(X|Z, Y(t))
= \sum_{k=1}^{\infty} \frac{1 - \mathbb{E}[E[X|Z]^{2k}]}{2k(2k - 1)\ln 2} (\mathbb{E}[E[X^*|Y^*(t)]^{2k}] - \mathbb{E}[E[X^*|Y^*(t')]^{2k}])
\leq \sum_{k=1}^{\infty} \frac{1 - \mathbb{E}[E[X|Z]^2]}{2k(2k - 1)\ln 2} (\mathbb{E}[E[X^*|Y^*(t)]^{2k}] - \mathbb{E}[E[X^*|Y^*(t')]^{2k}])
= \mathcal{H} \sqrt{\mathbb{E}[E[X|Z]^2]}(t') - \mathcal{H} \sqrt{\mathbb{E}[E[X|Z]^2]}(t)
= \mathcal{H} \sqrt{1 - \text{mmse}(X|Z)}(t') - \mathcal{H} \sqrt{1 - \text{mmse}(X|Z)}(t),
$$

where $\mathcal{H}_\mu(t)$ is defined in Lemma \cite{18}. Since $\mathcal{H}_\mu(t)$ is absolutely continuous, it follows by the above argument that, whenever both derivatives exist, we have

$$
\frac{d}{dt} H(X|Z, Y(t)) \leq \mathcal{H}' \sqrt{1 - \text{mmse}(X|Z)}(t).
$$

To establish \cite{12}, we start again from \cite{31} and observe that

$$
H(X|Z, Y(t + \delta)) - H(X|Z, Y(t))
\geq (1 - \mathbb{E}[E[X|Z]^2]) \sum_{k=1}^{\infty} \frac{1}{2k(2k - 1)\ln 2} (\mathbb{E}[E[X^*|Y^*(t)]^{2k}] - \mathbb{E}[E[X^*|Y^*(t + \delta)]^{2k}])
= \text{mmse}(X|Z)(\mathcal{H}(t + \delta) - \mathcal{H}(t)),
$$

where the inequality follows from $\mathbb{E}[E[X|Z]^{2k}] \leq \mathbb{E}[E[X|Z]^2]$ and $\mathbb{E}[E[X^*|Y^*(t)]^{2k}] \geq \mathbb{E}[E[X^*|Y^*(t + \delta)]^{2k}]$. Dividing both sides by $\delta$ and taking limit as $\delta \to 0$, we see that the derivative is lower bounded by $\text{mmse}(X|Z)\mathcal{H}'(t)$ whenever both derivatives exist, which is almost everywhere because they are both absolutely continuous. This completes the proof of \cite{10}.

To prove \cite{12}, we first define

$$
\psi(t) \triangleq H(X|Y(t), Z_2) - H(X|Y(t), Z_1)
= \frac{1}{\ln 2} \sum_{k=1}^{\infty} \frac{1}{2k(2k - 1)} (1 - \mathbb{E}[E[X^*|Y^*(t)]^{2k}]) (\mathbb{E}[E[X|Z_1]^{2k}] - \mathbb{E}[E[X|Z_2]^{2k}]).
$$

Next, we compute

$$
\psi(t + \delta) - \psi(t)
= \frac{1}{\ln 2} \sum_{k=1}^{\infty} \frac{1}{2k(2k - 1)} (\mathbb{E}[E[X^*|Y^*(t)]^{2k}] - \mathbb{E}[E[X^*|Y^*(t + \delta)]^{2k}]) (\mathbb{E}[E[X|Z_1]^{2k}] - \mathbb{E}[E[X|Z_2]^{2k}])
\geq \frac{1}{2\ln 2} (\mathbb{E}[E[X^*|Y^*(t)]^{2}] - \mathbb{E}[E[X^*|Y^*(t + \delta)]^{2}]) (\mathbb{E}[E[X|Z_1]^2] - \mathbb{E}[E[X|Z_2]^2]).
$$
= \frac{1}{2\ln 2} (\mathcal{M}(t + \delta) - \mathcal{M}(t)) (\text{mmse}(X|Z_2) - \text{mmse}(Z|Z_1)) ,

where each term in the first sum is non-negative because degradation (e.g., see Lemma 41) implies

\[ E \left[ |X|^2 \right] \geq E \left[ |X|^2(t + \delta) \right] \]

\[ E \left[ |X|^2 \right] \geq E \left[ |X|^2(t) \right] , \]

and the inequality follows from keeping only the \( k = 1 \) term. From above, we know that \( \psi(t) \) is absolutely continuous and, thus, \( \psi'(t) \) exists for almost all \( t \in [0,1] \). Dividing both sides by \( \delta \) and taking limit as \( \delta \to 0 \), we see that \( \psi'(t) \geq \mathcal{M}'(t) (\text{mmse}(X|Z_2) - \text{mmse}(X|Z_1)) \) whenever both derivatives exist, which is almost everywhere.

**Proof of Lemma 27.** We can start by writing

\[ G_i(t) = \frac{\partial}{\partial s_i} H(X \mid Y(s_0, \ldots, s_{N-1})) \bigg|_{s_0 = \cdots = s_{N-1} = t} = \frac{\partial}{\partial s_i} (H(X_i \mid Y(s_0, \ldots, s_{N-1})) + H(X_{\sim i} \mid Y(s_0, \ldots, s_{N-1}, X_i))) \bigg|_{s_0 = \cdots = s_{N-1} = t} = \frac{\partial}{\partial s_i} H(X_i \mid Y(s_i), Y_{\sim i}(t)) \bigg|_{s_i = t} , \]

(32)

where \( \frac{\partial}{\partial s_i} H(X_{\sim i}|Y(s_0, \ldots, s_{N-1}), X_i) = 0 \) because it is independent of \( s_i \). Then, Lemma 20 gives the first stated result. In particular, Lemma 20 implies that \( G_i(t) \) exists almost everywhere on \( [0,1] \) and that (10) can be used to bound it.

Combining (32) and Lemma 20, we will now bound the GEXIT function. First, we apply (32) by writing (10) with the GEXIT substitutions \( X \mapsto X_i, Y^*(\cdot) \mapsto Y_i(\cdot), \) and \( Z \mapsto Y_{\sim i}(t) \). Replacing \( \text{mmse}(X|Y_{\sim i}(t)) \) by \( M_i(t) \) gives the bound stated in (13). Applying the same substitutions to (11) gives (14).

### 6.3 Main Results

**Proof of Lemma 30.** Starting with (25), we can add and subtract terms with \( i = j \) to get

\[ \sum_{i,j \in [N]: i \neq j} (G_i(t) - G_j(t)) = \sum_{j \in [N]} \left( \sum_{i \in [N]} (G_i(t) - G_j(t)) \right) + \sum_{i \in [N]} (G_i(t) - G_i(t)) . \]

(33)

From the definition of the GEXIT function and the law of total derivative, the first summation over \( i \) on the RHS can be expressed as

\[ \sum_{i \in [N]} (G_i(t) - G_i(t)) = \frac{d}{dt} \left( H(X \mid Y(t)) - H(X \mid Y(t), Y'_i(t)) \right) \]

where \( Y'_i(t) \) is an second look at the \( j \)-th entry. The assumptions on the channel imply that the conditional entropy terms are equal to zero at \( t = 0 \) and \( H(X) \) and \( t = 1 \), and so the integral of this term w.r.t \( t \in [0,1] \) vanishes.

Next, we consider the second term on the RHS of (33). The definition of the GEXIT function implies that

\[ G_i(t) - G_i(t) = \frac{\partial}{\partial s_i} \left( H(X \mid Y_i(s), Y'_i(s), Y_{\sim i}(t)) - H(X \mid Y_i(s), Y_{\sim i}(t)) \right) \bigg|_{s=t} = \frac{\partial}{\partial s_i} \left( H(X \mid Y_i(s), Y'_i(t), Y_{\sim i}(t)) - H(X \mid Y_i(s), Y_{\sim i}(t)) \right) \bigg|_{s=t} \]

33
the degradation ordering of the channel family and another factor is non-positive by Jensen’s inequality.

For each term in the summation, we observe that one factor is non-decreasing in $X_i$. Combining the area theorem (27) with the integral representation $\frac{d}{dt} H_i(t)$ gives the following decomposition

$$H(X_i | Y_i(t), Y_i^*(t), Y_{\sim i}(t)) = H(X_i | Y_i(t), Y_i^*(t), Y_{\sim i}(t)) - H(X_i | Y_i(t), Y_i^*(t), Y_{\sim i}(t)).$$

where we recall that $(X_i^*, Y_i^*(s))$ denotes an input-output pair from a single use of channel $i$ where $X_i^*$ is uniform. For each term in the summation, we observe that one factor is non-decreasing in $s$ due to the degradation ordering of the channel family and another factor is non-positive by Jensen’s inequality (or equivalently because removing the observation of $Y_i^*(t)$ gives a degraded observation of $X_i$). Thus, we can conclude that the $s$-derivative of each term is non-positive.

For the second term, we apply the two-look formula with respect to $Y_i(t)$ and $(Y_i^*(t), Y_{\sim i}(t))$ to obtain

$$\frac{d}{ds} H_i(t) = \frac{d}{dt} H_i(t) = \frac{d}{dt} H_i(t) = \frac{d}{dt} H_i(t).$$

where the second term on the second line is zero because $Y_i^*(s)$ is conditionally independent of $X_{\sim i}$ given $X_i$ and, on the third line, $Y_i^*(s)$ is transformed into $Y_i^*(s)$ by the two-look formula. In this case, one factor is again non-decreasing in $s$ (for the same reason as above) and another factor is bounded from above by one. Accordingly, we can upper bound the difference in GEXIT functions by the second term and this gives

$$G_i^*(t) - G_i(t) = \frac{d}{dt} H_i(t),$$

where $H_i$ is the entropy function of the $i$-th channel. Integrating both sides of this inequality over the unit interval gives an upper bound of $H_i(1) = 1$. This concludes the proof of (25).

If the input distribution has doubly-transitive symmetry, then $G_i^j = G_i^k$ for all $i, j, k, \ell \in [N]$ with $i \neq j$ and $k \neq \ell$. This implies that, in (25), all terms in the sum are equal. Thus, we can divide by $N(N-1)$ (i.e., the total number of terms) to see that each term satisfies (20). □

Proof of Lemma [71]. The existence and uniqueness of $t_R$ follow because $C(\cdot)$ is continuous and strictly increasing. Combining the area theorem [27] with the integral representation $C(t) = 1 - \mathcal{H}(t) =
\[ \int_t^1 \mathcal{H}'(s) \, ds \text{ leads to the following decomposition:} \]

\[ C(t) - R = - \int_0^t G(s) \, ds + \int_t^1 (\mathcal{H}'(s) - G(s)) \, ds. \] (34)

Notice that this difference is strictly negative on \([0, t_R]\) and strictly positive on \((t_R, 1]\). By the I-MMSE relations in Lemma \[21\], the integrands are related to the extrinsic MMSE function via the following inequalities, which hold almost everywhere on the unit interval:

\[ 0 \leq \mathcal{H}'(t) - G(t) \leq (1 - M(t))\mathcal{H}'(t) \]
\[ 0 \leq G(t) \leq \mathcal{H}'_{\sqrt{1 - M(t)}}(t). \]

Here, we recall that \(\mathcal{H}_\mu(\cdot)\) is the entropy function of the BMS channel when the input has mean \(\mu \in [-1, 1]\) (see Lemma \[18\]).

To prove the upper bound on \(M(t)\) we combine the upper bound on \(\mathcal{H}'(s) - G(s)\) with the lower bound on \(G(s)\) to obtain

\[ C(t) - R \leq \int_t^1 (1 - M(s)) \mathcal{H}'(s) \, ds. \]

Multiplying both sides by \(M(t)\) and recalling that \(M(\cdot)\) is non-decreasing allows us to write

\[ M(t)(C(t) - R) \leq \int_t^1 M(s)(1 - M(s)) \mathcal{H}'(s) \, ds \]
\[ \leq \kappa(t) \int_t^1 M(s)(1 - M(s)) \, ds. \]

If \(t < t_R\) then \(C(t) > R\) and we can divide both sides by \(C(t) - R\) to obtain \[28\]

The lower bound in \[29\] is obtained in two stages. First, we can multiply both sides of \[34\] by negative one and apply the upper bound on \(G(s)\) and the lower bound on \(\mathcal{H}'(s) - G(s)\) to obtain

\[ R - C(t) \leq \int_0^t \mathcal{H}'_{\sqrt{1 - M(s)}}(s) \, ds. \] (35)

Since \(M(s)\) is non-decreasing in \(s\) and \(\mathcal{H}'_\mu(s)\) is non-increasing in \(\mu^2\) (see Lemma \[18\]) we have

\[ \mathcal{H}'_{\sqrt{1 - M(s)}}(s) \leq \mathcal{H}'_{\sqrt{1 - M(t)}}(s), \quad 0 \leq s \leq t \leq 1. \]

Integrating both sides gives

\[ \int_0^t \mathcal{H}'_{\sqrt{1 - M(s)}}(s) \, ds \leq \int_0^t \mathcal{H}'_{\sqrt{1 - M(t)}}(s) \, ds \]
\[ \leq \int_0^1 \mathcal{H}'_{\sqrt{1 - M(t)}}(s) \, ds \]
\[ = h_b \left( \frac{1 - \sqrt{1 - M(t)}}{2} \right), \]

where we have use the fact that \(\mathcal{H}'_\mu(\cdot)\) is non-negative and \(\mathcal{H}_\mu(1) = h_b((1 - \mu)/2)\). The mapping \(z \mapsto h_b \left( \frac{1 - \sqrt{1 - z}}{2} \right)\) is strictly increasing on \([0, 1]\] with inverse given by \(\psi(\cdot)\). In combination with \[35\], this implies that

\[ M(t) \geq \psi(R - C(t)), \quad t \in [t_R, 1]. \]
We can strengthen this lower bound by incorporating knowledge about the area under the $M(s)(1-M(s))$ curve. To do this, we write

$$\int_{t_R}^{t} M(s)(1-M(s)) \, ds \geq (1-M(t)) \int_{t_R}^{t} M(s) \, ds$$

$$\geq (1-M(t)) \left( \int_{t_R}^{t} \psi(R-C(s)) \, ds \right).$$

Since $\psi(\cdot)$ is non-negative and strictly increasing, the integral is strictly positive and so we can rearrange terms to obtain the bound given in (29).

References


A Additional Material

A.1 BMS Channels with General Output Alphabets

For the purpose of our proof, it is convenient to focus on BMS channels satisfying the conditions in Definition 10, i.e., the output alphabet is equal to the extended reals and the transition probability satisfies \( w(y \mid +1) = w(-y \mid -1) \). In this section, we provide a more general definition of BMS channels with respect to an arbitrary output alphabet \( \mathcal{Y} \) and show any any channel satisfying this definition can be mapped to one satisfying the conditions of Definition 10.

Let \( W_{Y \mid X} \) be a binary channel with input alphabet \( \mathcal{X} = \{ \pm 1 \} \), output alphabet \( \mathcal{Y} \), and let \( w(y \mid x) \) denote the conditional density of \( Y \) with respect to a fixed dominating measure. It well-known that a minimal sufficient statistic for estimating \( X \) from \( Y \) is provided by log-likelihood ratio \( \ell : \mathcal{Y} \to \mathbb{R} \), which is defined by

\[
\ell(y) := \log \frac{w(y \mid +1)}{w(y \mid -1)}.
\]

Note that in cases where the output uniquely defines the input (e.g., the perfect channel), the log-likelihood ratio can take the values \( \pm \infty \) in the extended real numbers.

**Definition 39 (Channel Symmetry).** A binary channel \( W_{Y \mid X} \) with input alphabet \( \mathcal{X} = \{ \pm 1 \} \) and log-likelihood ratio \( \ell \) is called symmetric if the conditional distribution of \( \ell(Y) \) given the input is \( +1 \) is equal to the conditional distribution of \( -\ell(Y) \) given the input is \( -1 \).

For a symmetric channel, the relevant properties of the channel are completely summarized by the distribution of the log-likelihood ratio when the input is \( +1 \). This distribution is often referred to as the L-density of the channel \[31\]. As a consequence, the specific details of channel the output space \( \mathcal{Y} \) can be neglected and one may assume, without loss of generality, that the output alphabet is a subset of the extended reals. For example, if a random variable \( X \in \{ \pm 1 \} \) is transmitted through a symmetric binary channel \( W_{Y \mid X} \) that produces an output \( Y \), then the sufficient statistic \( \ell(Y) \) can be expressed as the product of the input \( X \) and an independent noise term \( Z \) according to:

\[
\ell(Y) = XZ
\]

where \( Z := X \ell(Y) \) is drawn according to the conditional distribution of \( \ell(Y) \) when the input is \( +1 \).

A.2 Degradation Ordering of Channels

This section reviews some facts about channel degradation. The basic idea is that a channel \( W_{Z \mid X} \) is degraded with respect to a channel \( W_{Y \mid X} \) if the output of \( W_{Z \mid X} \) can be simulated by post-processing the output of \( W_{Y \mid X} \).

**Definition 40 (Channel Degradation \[31\] p. 204).** Consider channels \( W_{Y \mid X} \) and \( W_{Z \mid X} \) defined on the same input alphabet \( \mathcal{X} \). The channel \( W_{Z \mid X} \) is said to be (stochastically) degraded with respect to \( W_{Y \mid X} \) if there exists a third channel \( W_{Z \mid Y} \) with input alphabet \( \mathcal{Y} \) and output alphabet \( \mathcal{Z} \) such that \( W_{Z \mid X} \) is equal to the composition of \( W_{Y \mid X} \) and \( W_{Z \mid Y} \). For example, if \( w_{Y \mid X}(y \mid x) \) is a probability density function this means that

\[
w_{Z \mid X}(z \mid x) = \int_{\mathcal{Y}} w_{Z \mid Y}(z \mid y) w_{Y \mid X}(y \mid x) \, dy, \quad x \in \mathcal{X}, z \in \mathcal{Z}
\]

and if \( w(y \mid x) \) is a probability mass function then the same expression holds with the integral replaced by a summation.

In some cases, the relationship between random variables is described without specifying the channel explicitly. If \( Y \) and \( Z \) represent two observations of a third random variable \( X \), we say that \( X \) is stochastically degraded w.r.t. \( Z \) if the channel \( W_{Z \mid X} \) is degraded w.r.t. the channel \( W_{Y \mid X} \).

The above definition is equivalent \[31\] p. 205] to the statement that, for any distribution \( p_X \) on the input alphabet \( \mathcal{X} \), there exists a joint distribution on random variables \( (X, Y, Z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \) such that:
• \(X\) has distribution \(p_X\).
• \(Y\) is an observation of \(X\) through channel \(W_{Y|X}\)
• \(Z\) is an observation of \(X\) through channel \(W_{Z|X}\); and
• \(X - Y - Z\) forms a Markov chain.

The following is closely related to previous characterizations of channel degradation \[31\] p. 206.

**Lemma 41** (Convex Order). Let \(X \in \mathcal{X}\) be a random variable that is transmitted through two channels \(W_{Y|X}\) and \(W_{Z|X}\) whose outputs are \(Y\) and \(Z\), respectively. If \(W_{Z|X}\) is degraded with respect to \(W_{Y|X}\), then for all convex functions \(\phi: \mathcal{X} \to \mathbb{R}\), we have

\[
E[\phi(E[X | Y])] \geq E[\phi(E[X | Z])]
\]

provided that the expectations exist. In particular, if \(X\) is real-valued then

\[
E\left[ E[X | Y]^{2k} \right] \geq E\left[ E[X | Z]^{2k} \right], \quad k \in \mathbb{N}.
\]

**Proof.** Observe that expectations in the inequality depend only on the marginal distributions of the pairs \((X, Y)\) and \((X, Z)\) and thus we are free to consider any joint distribution on \((X, Y, Z)\) with the same pairwise marginals. From the definition of channel degradation, there exists a joint distribution such that \(X - Y - Z\) forms a Markov chain. Under this distribution, the conditional expectation satisfies \(E[X | Y, Z] = E[X | Y]\) almost surely and so the first result follows from writing

\[
E[\phi(E[X | Y])] = E[\phi(E[X | Y, Z])]
\]

\[
= E[ E[\phi(E[X | Y, Z]) | Z]]
\]

\[
\geq E[\phi(E[E[X | Y, Z] | Z])]
\]

\[
= E[\phi(E[X | Z])],
\]

where the third step follows from Jensen’s inequality and the convexity of \(\phi\). The second result holds because \(\phi(x) = x^{2k}\) is convex on \(\mathbb{R}\) for all positive integers \(k\).

\[\square\]

### A.3 Comparison with Earlier Proof for the BEC

This section discusses the relationship between the approach used in this paper, which applies generally for any BMS channel, and approach used in earlier work \[7\] focusing on the special case of the BEC. Recall that the proof in this paper depends crucially the nesting property of RM codes described Section 2.2.

In comparison, the approach in \[7\] combines special properties of the BEC with results from the theory of boolean functions \[51, 52\] to prove that any sequence of codes with a doubly transitive symmetry group achieves capacity.

For the BEC, let \(t\) denote the erasure rate and recall that the GEXIT function simplifies to the EXIT function in this case. Thus, we have

\[G_i(t) = H(X_i | Y_{\sim i}(t)).\]

In addition, for any received sequence, the channel input \(X_i\) is either recoverable or unknown. It follows that the \(E[X_i|Y_{\sim i}(t)]^2 \in \{0, 1\}\) and the extrinsic MMSE also satisfies

\[M_i(t) = 1 - E \left[ E[X_i|Y_{\sim i}(t)]^2 \right] = \Pr \left( \text{E} \left[ X_i | Y_{\sim i}(t) \right] = 0 \right) = H(X_i | Y_{\sim i}(t)).\]

Now, we assume that the code has transitive symmetry so that we can restrict our attention to \(M(t) := M_0(t)\) and use Lemma 24 to upper bound the variance of the estimate.

Next, we will evaluate \(\Delta^j_0 := \Delta^j_0\) by starting from its definition in \[18\]. Suppressing \(t\), we can rewrite this as \(\Delta^j_0 = \frac{1}{2}E \left[ D_j(Y, Y_j') \right]\), where \(Y_j'\) is an independent observation of \(X_j\) through the same channel and

\[D_j(y, y') := E \left[ \left( \text{E}[X_0 | Y_{\sim 0}] - \text{E}[X_0 | Y_{\sim 0, j}, Y_j'] \right)^2 | Y = y, Y_j' = y' \right].\]
Next, we observe that $D_j(y, y') \in \{0, 1\}$ and it equals 0 unless $y_j \neq y'_j$. If $y_j \neq y'_j$, then this quantity is related to the influence (from the theory of boolean functions) and we see that

$$
E \left[ D_j(Y, Y'_j) \mid Y_j \neq Y'_j \right] = \Pr(E \left[ X_0 \mid Y_{-0} \right] \neq E \left[ X_0 \mid Y_{-0,j}, Y'_j, Y'_j \neq Y_j \right]) = I_j,
$$

where $I_j$ is influence of the $j$-th received value on the EXIT function as defined in [7]. Since we have $\Pr(Y_j \neq Y'_j) = 2t(1 - t)$, it follows that

$$
\Delta^j_0(t) = \frac{2t(1 - t)}{2} I_j(t). \quad (36)
$$

From [7], Remark 18, we also know that

$$
I_j(t) = \left\{ \frac{d}{ds_j} H(X_0 | Y_{-0}(s_0, \ldots, s_{N-1})) \right\}_{(s_0, \ldots, s_{N-1}) = (t, \ldots, t)}.
$$

Notice that $I_0(t) = 0$ because $Y_{-0}$ does not depend on $Y_0$. Assuming doubly transitive symmetry, we see that $I_j(t) = I_1(t)$ for all $j \in [N] \setminus \{0\}$. Thus, the total derivative formula implies that

$$
\frac{d}{dt} H(X_0 | Y_{-0}(t)) = (N - 1) I_1(t).
$$

Following the approach in this paper, we can use (36) to see that

$$
\int_0^1 \Delta^j_0(t) \, dt \leq \frac{1}{4} \int_0^1 I_j(t) \, dt = \frac{1}{4(N - 1)} \int_0^1 \left( \frac{d}{dt} H(X_0 | Y_{-0}(t)) \right) \, dt = \frac{1}{4(N - 1)},
$$

where the integral equals 1 if the minimum distance of the code is at least 2. We can also apply this bound to a subset $A \subseteq [N]$ by summing over all $j \in A$. From this, we see that the total contribution will vanish as long as $|A|/N$ vanishes for the chosen sequence of codes.

In contrast, the proof in [7] is based on results from the theory of boolean functions [51, 52] that imply $I_1(t) \geq C \frac{\ln N}{N} H(X_0 | Y_{-0}(t))$ for some constant $C > 0$. Thus, the proof in [7] shows that

$$
\frac{d}{dt} H(X_0 | Y_{-0}(t)) \geq \frac{N - 1}{N} C(\ln N) H(X_0 | Y_{-0}(t)).
$$

This implies that $H(X_0 | Y_{-0}(t))$ must transition from 0 to 1 over an interval whose width is roughly $1/(C \ln N)$.

In this paper, the remaining terms in (19) are grouped together. To analyze $C = \text{RM}(r, m)$ with $N = 2^m$, we choose $k \geq 1$ and define $A = [2^{m-k}]$. By Lemma 5, we see that $X_A$ is a uniform random codeword from $\text{RM}(r, m - k)$. Then, we define $B = [N] \setminus A$ and recall, from Section 5.2.2, that

$$
\Delta^B_0(t) = \frac{1}{2} \mathbb{E} \left[ (E[X_0 | Y_A(t), Y_B(t)] - E[X_0 | Y_A(t), Y'_B(t)])^2 \right],
$$

where $Y'(t)$ denotes an independent second observation of $X$ through a BEC with the same erasure probability. Since we are working on the BEC, both inner conditional expectations can only take values in the set $\{-1, 0, 1\}$ with 0 indicating erasure and ±1 indicating successful recovery. Thus, we can simplify $\Delta^B_0(t)$ by expanding the square and taking expectations to get

$$
\Delta^B_0(t) = E \left[ E[X_0 | Y_A(t), Y_B(t)]^2 \right] - E \left[ E[X_0 | Y_A(t), Y_B(t)] E[X_0 | Y_A(t), Y'_B(t)] \right]
\leq E \left[ E[X_0 | Y_A(t), Y_B(t)]^2 \right] - E \left[ E[X_0 | Y_A(t), Y_B(t), Y'_B(t)] \right]
= H(X_0 | Y_A(t), Y_B(t)) - H(X_0 | Y_A(t), Y_B(t), Y'_B(t))
\leq H(X_0 | Y_A(t), Y_B(t)) - H(X'_0 | Z_{-0}(t)).
$$

The first inequality holds because it may be possible to recover $X_0$ by jointly processing $Y_A, Y_B, Y'_B$ even when it cannot be recovered separately from either $Y_A, Y_B$ or $Y_A, Y'_B$. The second inequality follows from
assuming that $Z(t)$ is the observation of a uniform random codeword $X'$ from RM($r, m + k$) and that $(X_0, Y_A, Y_B, Y_B')$ is equal in distribution to $(X_0', Z_A, Z_B, Z_C)$ (e.g., see Lemma 8 and Section 5.2.2).

Finally, we can put things together. First, we can integrate the upper bound on $\Delta_0^R(t)$ to see that
\begin{align*}
\int_0^1 \Delta_0^R(t) \, dt & \leq \int_0^1 H(X_0|Y_A(t), Y_B(t)) \, dt - R(r, m + k) \\
& \leq \frac{3k + 4}{5\sqrt{m}},
\end{align*}
where the last step follows from Lemma 7. Then, we can integrate to see that
\begin{align*}
\int_0^1 M_i(t)(1 - M_i(t)) \, dt & \leq \int_0^1 \Delta_i^R(t) \, dt + \sum_{j \notin B} \int_0^1 \Delta_j^R(t) \, dt \\
& \leq \frac{3k + 4}{5\sqrt{m}} + \frac{2^{m-k}}{4(2^m - 1)}.
\end{align*}
This upper bound vanishes if we consider a code sequence where $m \to \infty$ with $k$ chosen according to $k = [\log_2 m]$. Thus, the EXIT function has a sharp threshold and the EXIT area theorem (e.g., see [7, Proposition 11]) implies that $M(t) = H(X_0|Z_{\infty}(t))$ will jump at $1 - R$ in the limit.

### A.4 Localization of Jump in Extrinsic MMSE via Sequences

In Section 5.3 we provide non-asymptotic bounds on the extrinsic MMSE associated with a family of BMS channels and an RM($r, m$) code. Applying these bounds to a sequence of RM codes with strictly increasing blocklength and code rate converging to $R \in (0, 1)$, shows that the extrinsic MMSE converges to a 0-1 step function that jumps at the unique point $t_R$ such that $C(t_R) = R$. For that result, this section provides an alternative proof which may be of independent interest.

In particular, we make use of Lemma 43 below which shows that convergence of the extrinsic MMSE to 0 or 1 is equivalent to convergence of the GEXIT to its lower and upper bounds, respectively.

Let $\{C^{(n)}\}_{n \in \mathbb{N}}$ be a sequence of transitive codes with strictly increasing blocklength and rate converging to $R \in (0, 1)$. For a BMS family satisfying Assumption 1 let $\{(C^{(n)}), M^{(n)}\}_{n \in \mathbb{N}}$ be the corresponding sequence of GEXIT functions and extrinsic MMSE functions. The bounds given here and in Section 5.3 depend primarily on the quantity
\[ a_n = \int_0^1 M^{(n)}(s)(1 - M^{(n)}(s)) \, ds. \]
We will see that a code sequence achieves capacity on the family of BMS channels if $a_n \to 0$.

The approach taken in this section is a proof by contradiction. Suppose that $a_n \to 0$ but the sequence of extrinsic MMSE functions, $M^{(n)}(t)$, does not converge to a 0-1 step function that jumps at $t = t_R$. Then, one of two things must happen. Either there is a $t' < t_R$, an $\epsilon \in (0, 1)$, and a subsequence $M^{(n_k)}(t)$ such that $M^{(n_k)}(t') \geq \epsilon$ for all $k \in \mathbb{N}$. Or, there is a $t' > t_R$, an $\epsilon \in (0, 1)$, and a subsequence $M^{(n_k)}(t)$ such that $M^{(n_k)}(t') \leq 1 - \epsilon$ for all $k \in \mathbb{N}$.

The following lemma implies that both of the possibilities lead to contradictions. To see this, we recall that the area theorem implies
\[ \int_0^1 C^{(n)}(t) \, dt \to R. \]
This also implies that the limit is the same for any subsequence $G^{(n_k)}(t)$. Now, for the $t' < t_R$ case, we see 37 implies that the limit inferior of the sequence of GEXIT integrals is at least $C(t') > R$ which gives a contradiction. For the $t' > t_R$ case, we see 38 and 13 together imply that the sequence of GEXIT integrals is upper bounded by $C(t') = \int_0^1 H'(t) \, dt < R$ which gives a contradiction.

The lemma is obtained by combining an upper bound on $a_n$ (e.g., see Lemma 27) with the comparison between the GEXIT function $G^{(n)}(t)$ and the extrinsic MMSE $M^{(n)}(t)$ established in Lemma 43. Thus,
the sequence of extrinsic MMSE functions, $M^{(n)}(s)$, converges to a 0-1 step function that jumps at $t = t_R$.

Finally, applying Lemma 43 again shows that the sequence of GEXIT functions $G^{(n)}(s)$ converges almost everywhere to a function that jumps from 0 to $\mathcal{H}'(t)$ at $t = t_R$.

**Lemma 42.** Under the assumptions stated above, if $a_n \to 0$, then, for every $t' \in (0,1)$, we have

\[
\liminf_{n \to \infty} M^{(n)}(t') > 0 \quad \implies \quad \int_{t'}^1 G^{(n)}(t) \, dt \to C(t')
\]

\[
\limsup_{n \to \infty} M^{(n)}(t') < 1 \quad \implies \quad \int_0^{t'} G^{(n)}(t) \, dt \to 0.
\]

**Proof.** If $\liminf_{n \to \infty} M^{(n)}(t') > 0$ then there exists an $\epsilon \in (0,1)$ and an integer $N$ such that $M^{(n)}(t') \geq \epsilon$ for all $n \geq N$. For $t \in (t',1]$ and $n \geq N$, we can write

\[
\int_0^1 M^{(n)}(s)(1 - M^{(n)}(s)) \, ds \geq \int_{t'}^1 M^{(n)}(s)(1 - M^{(n)}(s)) \, ds \geq \epsilon(t - t')(1 - M^n(t)),
\]

where the second inequality follows from $\epsilon \leq M^{(n)}(s) \leq M^{(n)}(t)$ for $s \in [t',t]$. By assumption, the LHS (i.e., $a_n$) converges to 0 and this proves that $M^{(n)}(t) \to 1$ for all $t \in (t',1]$. By Lemma 43 it follows that

\[
\int_{t'}^1 (\mathcal{H}'(t) - G^{(n)}(t)) \, dt \to 0,
\]

which is equivalent to the stated result in view of the fact that $C(t') = \int_{t'}^1 \mathcal{H}'(t) \, dt$.

For the second statement, the argument is essentially the same. If $\limsup_{n \to \infty} M^{(n)}(t') < 1$ then there exists an $\epsilon \in (0,1)$ and an integer $N$ such that $M^{(n)}(t') \leq 1 - \epsilon$ for all $n \geq N$. For $t \in [0,t')$ and $n \geq N$, we can write

\[
\int_0^1 M^{(n)}(s)(1 - M^{(n)}(s)) \, ds \geq \int_t^{t'} M^{(n)}(s)(1 - M^{(n)}(s)) \, ds \geq M^{(n)}(t')(t' - t)\epsilon,
\]

where the second inequality follows from $1 - M^{(n)}(s) \geq 1 - M^{(n)}(t') \geq \epsilon$ for $s \in [t,t']$. By assumption, the LHS (i.e., $a_n$) converges to 0 and this proves that $M^{(n)}(t) \to 0$ for all $t \in [t',t)$. Similarly, the second result follows from applying Lemma 43. 

**Lemma 43.** Using the setup from Lemma 27, assume that $M(t)$ is strictly increasing and consider a sequence of problems where the BMS channel family is fixed but distribution on $X$ is changing. Let \( \{G^{(n)}, M^{(n)}\}_{n \in \mathbb{N}} \) be the corresponding sequence of GEXIT and extrinsic MMSE functions for the same symbol (say $X_0$). Then, for any $t' \in (0,1)$, we have

\[
\int_0^{t'} G^{(n)}(s) \, ds \to 0 \quad \iff \quad M^{(n)}(s) \to 0 \text{ for all } s \in [0,t')
\]

\[
\int_{t'}^1 (\mathcal{H}'(s) - G^{(n)}(s)) \, ds \to 0 \quad \iff \quad M^{(n)}(s) \to 1 \text{ for all } s \in (t',1].
\]

**Proof.** Without loss of generality, we assume that $G^{(n)}(t)$ and $M^{(n)}(t)$ are the GEXIT and extrinsic MMSE functions of $X_0$ for the $n$-th problem in the sequence. We will start by applying [13] and writing the results without the subscript $i$. Subtracting $\mathcal{H}'(t)$ from all the terms in [13] and then negating all terms shows that

\[
(1 - M^{(n)}(t)) \mathcal{H}'(t) \geq \mathcal{H}'(t) - G^{(n)}(t) \geq \frac{(1 - M^{(n)}(t))M'(t)}{2 \ln 2}
\]

almost everywhere on $[0,1]$. Since $1 - M^{(n)}(s)$ is non-increasing, it follows that $1 - M^{(n)}(t') \leq 1 - M^{(n)}(s) \leq 1 - M^{(n)}(t)$ for $s \in [t,t']$ and $0 \leq t \leq t' \leq 1$. Integrating (41) over the interval $[a,b]$ shows that

\[
(1 - M^{(n)}(b)) \frac{M(b) - M(a)}{2 \ln 2} \leq \int_a^b (\mathcal{H}'(s) - G^{(n)}(s)) \, ds \leq (1 - M^{(n)}(a))(\mathcal{H}(b) - \mathcal{H}(a)).
\]
Lemma 18. To complete the proof, for any inequality follows from the fact that where the second inequality follows from the fact that where the last step follows from two applications of (42). By the continuity of \( H(\cdot) \), for any \( \epsilon > 0 \), there exists \( t \in (t', 1) \) such that \( H(t) - H(t') < \epsilon \). Since \( M^{(n)}(t) \to 1 \), there exists \( N \in \mathbb{N} \) such that \( 1 - M^{(n)}(t) < \epsilon \) for all \( n > N \). Thus, the RHS converges to 0 because, for any \( \epsilon > 0 \), there is an \( N \in \mathbb{N} \) such that the RHS is less than \( 2\epsilon \) for all \( n > N \).

Proof of \( \iff \) in (40): Consider the left-hand inequality of (42). Since the integrand is non-negative, the integral from \( a = t' \) to \( b = t' + \delta \), with \( \delta \in (0, 1 - t'] \), is upper bounded by the integral from \( a = t' \) to \( b = 1 \). Thus, for all \( \delta \in (0, 1 - t'] \), we see that
\[
M^{(n)}(t' + \delta) \geq 1 - \frac{2\ln^2 M^{(n)}(t')}{\mathcal{M}(t' + \delta) - \mathcal{M}(t')} \int_{t'}^{t} (H'(s) - G^{(n)}(s)) \, ds.
\]
Since the integral on the RHS converges to 0 and \( \mathcal{M}(t + \delta) - \mathcal{M}(t) > 0 \), it follows that \( M^{(n)}(t' + \delta) \to 1 \) for all \( \delta \in (0, 1 - t'] \).

Proof of \( \implies \) in (39): First, for any \( \delta \in (0, t'] \), we can integrate the left-hand inequality of (13) over the interval \( [t' - \delta, t'] \) and lower bound to see that
\[
\int_{0}^{t'} G^{(n)}(s) \, ds \geq \int_{t' - \delta}^{t'} G^{(n)}(s) \, ds \geq \int_{t' - \delta}^{t'} M^{(n)}(s) H'(s) \, dt \geq (H(t') - H(t' - \delta)) M^{(n)}(t' - \delta)
\]
because \( G^{(n)}(s) \geq 0 \) almost everywhere and \( M^{(n)}(s) \geq M^{(n)}(t' - \delta) \) for \( s \in [t' - \delta, t'] \). Since \( \mathcal{M}(t) \) is strictly increasing, it implies that \( H(t') - H(t' - \delta) \geq (\mathcal{M}(t') - \mathcal{M}(t' - \delta))/(2 \ln 2) > 0 \) for \( \delta \in (0, t'] \).

Thus, if \( \int_{0}^{t'} G^{(n)}(s) \, ds \to 0 \), then \( M^{(n)}(t' - \delta) \to 0 \) for all \( \delta \in (0, t'] \).

Proof of \( \iff \) in (39): Since \( M^{(n)}(t) \) is non-decreasing and Lemma 18 establishes that \( \mathcal{H}'(t) \) is non-increasing in \( \mu \), we can upper bound the integral of (14) over the interval \([0, t]\) with
\[
\int_{0}^{t} G^{(n)}(s) \, ds \leq \int_{0}^{t} \frac{H'(s)}{1 - M^{(n)}(s)} \, ds
\]
\[
\quad \leq \int_{0}^{t} \frac{H'(s)}{1 - M^{(n)}(t)} \, ds
\]
\[
\quad \leq \int_{0}^{t} \frac{H'(s)}{1 - M^{(n)}(t)} \, ds
\]
\[
\quad = \frac{h \left( 1 - \sqrt{1 - M^{(n)}(t)} \right)}{2},
\]
where the second inequality follows from the fact that \( \mathcal{H}_{\sqrt{1 - M}}(t) \) is non-decreasing in \( m \), the third inequality follows from the fact that \( \mathcal{H}'(t) \geq 0 \) for all \( \mu, t \in [0, 1] \), and the final equality follows from Lemma 18. To complete the proof, for any \( t \in [0, t'] \), we write
\[
\int_{0}^{t'} G^{(n)}(s) \, ds = \int_{0}^{t} G^{(n)}(s) \, ds + \int_{t}^{t'} G^{(n)}(s) \, ds
\]
\[
\leq \frac{h \left( 1 - \sqrt{1 - M^{(n)}(t)} \right)}{2} + (H(t') - H(t)).
\]
By the continuity of $H(\cdot)$, for any $\epsilon > 0$, there exists $t \in [0, t')$ such that $H(t') - H(t) < \epsilon$. Since $M^{(n)}(t) \to 0$, continuity of the $h_b$-term in $M^{(n)}(t)$ implies that there is an $N \in \mathbb{N}$ such that it is less than $\epsilon$ for all $n > N$. Thus, the RHS converges to 0 because, for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that the RHS is less than $2\epsilon$ for all $n > N$. \qed