Spatial Coupling, Potential Functions, and the Maxwell Construction

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Outline

- Coupled scalar recursions
- Simple proof of threshold saturation
- Extension to entire curve (NEW)
- Examples
Problem Setup

Let $f : \mathcal{X} \rightarrow \mathcal{X}$ and $g : \mathcal{X} \rightarrow \mathcal{X}$ be non-decreasing Lipschitz continuous functions on $\mathcal{X} = [0, x_{\text{max}}] \subseteq \mathbb{R}$. This talk will describe how the dynamics of the scalar recursion (from $x^{(0)} = x_{\text{max}}$)

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\begin{align*}
    y^{(\ell+1)} &= g \left( x^{(\ell)} \right) \\
    x^{(\ell+1)} &= f \left( y^{(\ell+1)} \right)
\end{align*}
$$

gives the fixed point of the coupled recursion (from $x^{(0)}_i = x_{\text{max}}$)

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\begin{align*}
    y^{(\ell+1)}_i &= g \left( x^{(\ell)}_i \right) \\
    x^{(\ell+1)}_i &= \sum_{j=1}^{M-w+1} A_{j,i} f \left( \sum_{k=1}^{M} A_{j,k} y^{(\ell+1)}_k \right) (i = 1, \ldots, M)
\end{align*}
$$

(scalar notation)
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$$y^{(\ell+1)} = g(x^{(\ell)})$$

$$x^{(\ell+1)} = A^\top f(A y^{(\ell+1)})$$

(vector notation)
A Few Details

Moving average of \( w \) values defined by \( A_{j,k} \triangleq [A]_{j,k} \) with

\[
A = \frac{1}{w} \begin{bmatrix}
1 & 1 & \ldots & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & \ddots & 1 & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & 1 & 1 & \ldots & 1
\end{bmatrix}
\]

Monotonicity and continuity of \( f, g \) imply:

- convergence to fixed points
- scalar case: \( x^{(\ell)} \downarrow x^{(\infty)} \) and \( y^{(\ell)} \downarrow y^{(\infty)} \)
- vector case: \( x_i^{(\ell)} \downarrow x_i^{(\infty)} \) and \( y_i^{(\ell)} \downarrow y_i^{(\infty)} \) for \( i = 1, \ldots, M \)

Q: What can we say about the coupled fixed point \( x^{(\infty)} \)?
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Moving average of $w$ values defined by $A_{j,k} \triangleq [A]_{j,k}$ with

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Q: What can we say about the coupled fixed point $x_i^{(\infty)}$?
Let the potential function $U_s: \mathcal{X} \to \mathbb{R}$ of the scalar recursion be

$$U_s(x) \triangleq x g(x) - G(x) - F(g(x)),$$

where $F(x) = \int_0^x f(z)dz$ and $G(x) = \int_0^x g(z)dz$.

Derivative of $U_s(x)$ describes the dynamics

$$\frac{d}{dx} U_s(x) = (x - f(g(x))) g'(x)$$
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**Theorem (YJNP)**

If \( f(g(x)) < x \) for \( x \in (0, u) \) and \( \min_{x \in \mathcal{X} \setminus [0,u]} U_s(x) > 0 \), then \( \exists w_0 < \infty \): for \( w > w_0 \), only fixed point of coupled recursion is \( x^{(\infty)} = 0 \)

- For LDPC DE, equals conjectured condition for MAP decoder
History of Threshold Saturation Proofs

For:

- the BEC by KRU in 2010
  - Established **many properties and tools** used by later approaches
- the Curie-Weiss model in physics by HMU in 2010
- CDMA using a GA by TTK in 2011
- CDMA with outer code via GA by Truhachev in 2011
- compressed sensing using a GA by DJM in 2011
- regular codes on BMS channels by KRU in 2012
- monotonic scalar and vector recursions by YJNP in 2012
- irregular LDPC codes on BMS channels by KYMP in 2012
- general scalar recursions by KRU in 2012
Simple Proof

Outline:

1. Define coupled-system potential function $U_c : \mathcal{X}^M \rightarrow \mathbb{R}$

$$U_c(x) = x^\top g(x) - G(x) - F(Ag(x))$$
Simple Proof

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3. Show that, if $x^{(\infty)} \neq 0$, then shifting $x$ towards the free boundary reduces $U_c(x)$ by a positive constant ind. of $w$
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5. **Contradiction implies that** $x^{(\infty)} = 0$
For $w > w_0$, the coupled fixed point satisfies

$$\max_{i \in \{1, \ldots, M\}} x_i^{(\infty)} \leq \bar{x}^* \triangleq \max \left( \arg \min_{x \in \mathcal{X}} U_s(x) \right)$$

For all $w$, the coupled fixed point satisfies

$$\max_{i \in \{1, \ldots, M\}} x_i^{(\infty)} \geq \bar{x}^* \triangleq \min \left( \arg \min_{x \in \mathcal{X}} U_s(x) \right) - \kappa \left( \frac{w-1}{M-w+1} \right)$$

$$\lim_{t \to 0} \kappa(t) = 0$$
Example: BEC Density Evolution of an LDPC Ensemble

\[ U(x; \varepsilon) = \varepsilon \lambda(y) \]
\[ \lambda(x) = \frac{4}{20} x + \frac{5}{20} x^2 + \frac{2}{20} x^6 + \frac{9}{20} x^{20} \]
\[ g(x; \varepsilon) = 1 - \rho(1 - x) \]
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That Looks Familiar

$$U(x;\varepsilon) = 10^{-2} \cdot 10^\varepsilon x$$
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\[ U(x; \varepsilon) = \varepsilon x \]
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\[ U(x; \varepsilon) \]

\[ \varepsilon \]

\[ x \]

\[ 0 \cdot 10^{-2} \]

\[ 3 \]

\[ -3 \]

\[ -6 \]
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\[
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Spatial Coupling, Potential Functions, and the Maxwell Construction
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\[ U(x; \varepsilon) \]

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\[ x \]

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\[ U(x; \varepsilon) = 3 \]

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\[ U(x; \varepsilon) = 9 \]

\[ U(x; \varepsilon) = 12 \]

\[ U(x; \varepsilon) = 15 \]

\[ U(x; \varepsilon) = 18 \]

\[ U(x; \varepsilon) = 21 \]

\[ U(x; \varepsilon) = 24 \]

\[ U(x; \varepsilon) = 27 \]
That Looks Familiar

\[ U(x; \varepsilon) = \frac{1}{10} \cdot 10^{3 \cdot \varepsilon} \]

\[ \varepsilon \]

\[ U(x; \varepsilon) \]

\[ x \]

\[ \varepsilon \]

\[ 0.2 \]

\[ 0.4 \]

\[ 0.6 \]

\[ 0.8 \]

\[ 1 \]

\[ 0.6 \cdot 10^{3 \cdot \varepsilon} \]

\[ 0.7 \]

\[ 0.8 \]

\[ 0.9 \]

\[ 10 \]
That Looks Familiar

\[ x \]

\[ \varepsilon \]
That Looks Familiar

\[ \text{EBP Fixed Points} \]
\[ \max_i x_i^{(\infty)} \text{ via DE} \]
\[ \arg \min_{x \in \mathcal{X}} U(x; \varepsilon) \]
Example: Analysis of an LDGM Ensemble

\[ f(y; \varepsilon) = \lambda(y) \]
\[ g(x; \varepsilon) = 1 - (1 - \varepsilon)\rho(1 - x) \]
\[ \lambda(x) = x^5 \]
\[ \rho(x) = \frac{2}{45} + \frac{2}{45}x + \frac{7}{15}x^2 + \frac{4}{9}x^3 \]
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Sketch of Proof for New Result

- Proof of new upper bound similar to “simple proof”
  - Only modified (i.e., one-sided) coupled recursion is changed
  - Vector values $\lt \bar{x}^*$ are increased to $\bar{x}^*$ after each iteration
  - Shift bound lemma refined to
    $$U_c(Sx) - U_c(x) \leq U_s(\bar{x}^*) - U_s([x]_N)$$

- Proof of new lower bound is based on a few observations
  - Initializing recursion to $\bar{x}^*$ lower bounds coupled fixed point
  - Iterations only decrease the potential
  - But, initial potential value implies $\max_i x_i^{(\infty)} \geq \bar{x}^* - o(1)$
Dependence on a Parameter

- **Family of admissible recursions** increasing in $\varepsilon \in \mathcal{E} = [0, \varepsilon_{\text{max}}]$
  - Scalar recursion defined by $f, g : \mathcal{X} \times \mathcal{E} \to \mathcal{X}$ with
    \[
    x^{(\ell+1)} = f \left( y^{(\ell+1)}; \varepsilon \right) \quad y^{(\ell+1)} = g \left( x^{(\ell)}; \varepsilon \right)
    \]
  - Scalar potential function $U_s : \mathcal{X} \times \mathcal{E} \to \mathbb{R}$ defined by
    \[
    U_s(x; \varepsilon) = x g(x; \varepsilon) - G(x; \varepsilon) - F(g(x; \varepsilon); \varepsilon)
    \]
  - Our new result bounds coupled fixed point as a function of $\varepsilon$
    \[
    \bar{x}^*(\varepsilon) \triangleq \max \{ x \in \mathcal{X} \mid U_s(x; \varepsilon) = \Psi(\varepsilon) \} \quad \Psi(\varepsilon) \triangleq \min_{x \in \mathcal{X}} U_s(x; \varepsilon)
    \]
The Maxwell Construction (1)

- Under mild conditions, the envelope theorem says that

\[
\frac{d}{d\epsilon} \Psi(\epsilon) = \frac{d}{d\epsilon} \min_{x \in X} U_s(x; \epsilon) \overset{a.e.}{=} U_s^{(0,1)}(\bar{x}^*(\epsilon); \epsilon)
\]

- Proof sketch: derivative of minimum depends on location \( \bar{x}^*(\epsilon) \) and \( \epsilon \) but the location term is zero due to minimum

- Computing \( U_s^{(0,1)}(x; \epsilon) \overset{\Delta}{=} \frac{d}{d\epsilon} U_s(x; \epsilon) \) shows that

\[
\Psi(\epsilon) = -\int_0^\epsilon \left( G^{(0,1)}(\bar{x}^*(t); t) + F^{(0,1)}(g(\bar{x}^*(t); \epsilon); t) \right) dt
\]

- For LDPC codes, we get \(-\frac{1}{L'(1)}\) times the MAP EXIT integral

\[
\Psi(\epsilon) = -\frac{1}{L'(1)} \int_0^\epsilon L(1 - \rho(1 - \bar{x}^*(t))) dt
\]
The curve $\Psi(\varepsilon) = \min_{x \in \mathcal{X}} U_s(x; \varepsilon)$ is Lipschitz continuous.

The curve $\bar{x}^*(\varepsilon)$ only jumps when the above minimum is achieved at multiple $x$ values.

Consider two ends of a $\bar{x}^*(\varepsilon)$ jump discontinuity:

- They must have the same value of the potential.
- If smooth fixed-point curve connects them, the integral along fixed-point curve must be zero.
- This is equivalent to the Maxwell construction.
Conclusions

- We analyze coupled scalar recursions
  - Coupled fixed point given by minimizer of scalar potential
  - Extends “saturation” from threshold to Maxwell curve
  - Valuable for systems with trivial perfect decoding thresholds
    - For example, LDGM codes have $x_i^{(\infty)} \to 0$ only if $\varepsilon \to 0$

- Dependence on a parameter easily incorporated
  - Min-potential curve $\Psi(\varepsilon) = \min_{x \in x} U_s(x; \varepsilon)$ of SC system is analogous to the “negative BP conditional entropy”
  - If smooth fixed-point curve connects discontinuities, then Maxwell construction gives the $x^*(\varepsilon)$ curve, which is analogous to the Maxwell or “conjectured MAP” curve