Spatial Coupling, Potential Functions, and the Maxwell Construction

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UC San Diego

Outline

- Coupled scalar recursions
- Simple proof of threshold saturation
- Extension to entire curve (NEW)
- Examples

Problem Setup

Let $f\colon \mathcal{X} \to \mathcal{X}$ and $g\colon \mathcal{X} \to \mathcal{X}$ be non-decreasing Lipschitz continuous functions on $\mathcal{X} = [0, x_{\max}] \subseteq \mathbb{R}$. This talk will describe how the dynamics of the scalar recursion (from $x^{(0)} = x_{\max}$)

$$y^{(\ell+1)} = g\left(x^{(\ell)}\right)$$
$$x^{(\ell+1)} = f\left(y^{(\ell+1)}\right)$$

gives the fixed point of the coupled recursion (from $x_i^{(0)} = x_{\max}$)

$$y_i^{(\ell+1)} = g\left(x_i^{(\ell)}\right)$$
$$x_i^{(\ell+1)} = \sum_{j=1}^{M-w+1} A_{j,i}\, f\left(\sum_{k=1}^{M} A_{j,k}\, y_k^{(\ell+1)}\right)^{(i = 1, \ldots, M)}$$

(scalar notation)

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gives the fixed point of the coupled recursion (from $\boldsymbol{x}^{(0)} = \boldsymbol{x}_{\max}$)

$$\begin{aligned} \boldsymbol{y}^{(\ell+1)} &= \boldsymbol{g}\left(\boldsymbol{x}^{(\ell)}\right) \\ \boldsymbol{x}^{(\ell+1)} &= \boldsymbol{A}^{\top} \boldsymbol{f}\left(\boldsymbol{A}\, \boldsymbol{y}^{(\ell+1)}\right) \end{aligned}$$

(vector notation)

A Few Details

- Moving average of w values defined by $A_{j,k} \triangleq [\boldsymbol{A}]_{j,k}$ with

$$\boldsymbol{A} = \frac{1}{w}\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \ddots & 1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

- Monotonicity and continuity of f, g imply:
  - convergence to fixed points
  - scalar case: $x^{(\ell)} \searrow x^{(\infty)}$ and $y^{(\ell)} \searrow y^{(\infty)}$
  - vector case: $x_i^{(\ell)} \searrow x_i^{(\infty)}$ and $y_i^{(\ell)} \searrow y_i^{(\infty)}$ for $i = 1, \ldots, M$

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- Q: What can we say about the coupled fixed point $x_i^{(\infty)}$?

Threshold Saturation

Let the potential function $U_\mathrm{s}\colon \mathcal{X} \to \mathbb{R}$ of the scalar recursion be

$$U_\mathrm{s}(x) \triangleq xg(x) - G(x) - F(g(x)),$$

where $F(x) = \int_0^x f(z)\mathrm{d}z$ and $G(x) = \int_0^x g(z)\mathrm{d}z$.

Derivative of $U_\mathrm{s}(x)$ describes the dynamics

$$\frac{\mathrm{d}}{\mathrm{d}x} U_\mathrm{s}(x) = \big(x - f\left(g(x)\right)\big) g'(x)$$

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Theorem (YJNP)

If $f(g(x)) < x$ for $x \in (0, u)$ and $\min_{x \in \mathcal{X} \setminus [0,u)} U_{\mathrm{s}}(x) > 0$, then $\exists w_0 < \infty$:

for $w > w_0$, only fixed point of coupled recursion is $\boldsymbol{x}^{(\infty)} = \mathbf{0}$

- For LDPC DE, equals conjectured condition for MAP decoder

History of Threshold Saturation Proofs

For:

- the BEC by KRU in 2010
    - Established many properties and tools used by later approaches
- the Curie-Weiss model in physics by HMU in 2010
- CDMA using a GA by TTK in 2011
- CDMA with outer code via GA by Truhachev in 2011
- compressed sensing using a GA by DJM in 2011
- regular codes on BMS channels by KRU in 2012
- monotonic scalar and vector recursions by YJNP in 2012
- irregular LDPC codes on BMS channels by KYMP in 2012
- general scalar recursions by KRU in 2012

Simple Proof

Outline:

1. Define coupled-system potential function $U_{\mathrm{c}} \colon \mathcal{X}^M \to \mathbb{R}$

$$U_{\mathrm{c}}(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{g}(\boldsymbol{x}) - G(\boldsymbol{x}) - F(\boldsymbol{A}\boldsymbol{g}(\boldsymbol{x}))$$

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4. But, Taylor expansion of the vector potential at fixed point shows potential change due to shift must vanish as $w \to \infty$
5. Contradiction implies that $\boldsymbol{x}^{(\infty)} = \mathbf{0}$

New Result

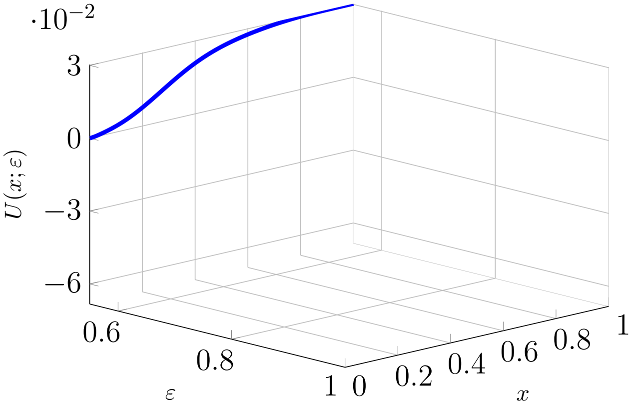
- For $w > w_0$, the coupled fixed point satisfies

$$\max_{i\in\{1,\ldots,M\}} x_i^{(\infty)} \leq \overline{x}^* \triangleq \max\left(\arg\min_{x\in\mathcal{X}} U_{\mathrm{s}}(x)\right)$$

- For all w, the coupled fixed point satisfies

$$\max_{i\in\{1,\ldots,M\}} x_i^{(\infty)} \geq \underline{x}^* \triangleq \min\left(\arg\min_{x\in\mathcal{X}} U_{\mathrm{s}}(x)\right) - \underbrace{\kappa\left(\tfrac{w-1}{M-w+1}\right)}_{\lim_{t\to 0}\kappa(t)=0}$$

Example: BEC Density Evolution of an LDPC Ensemble



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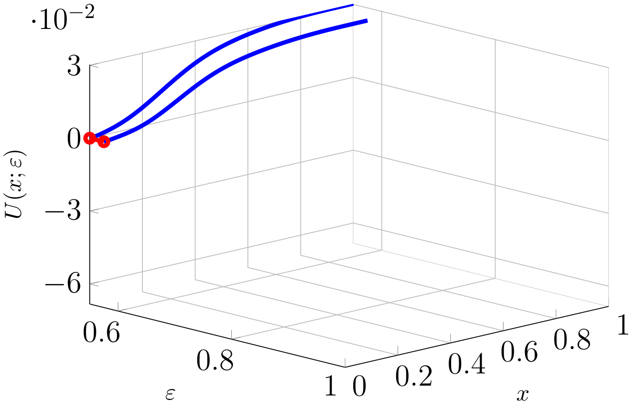

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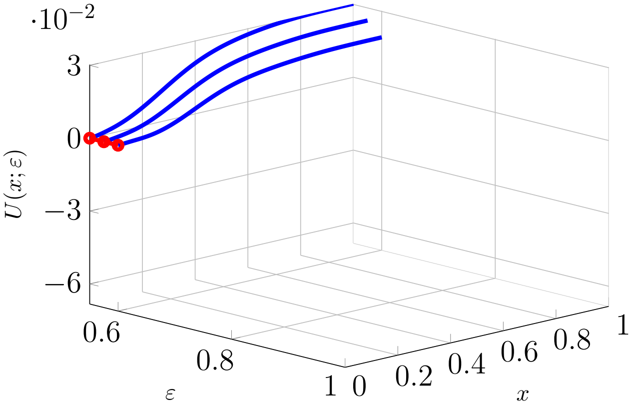

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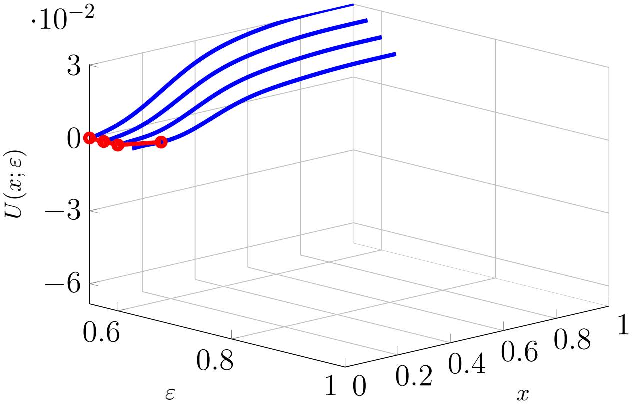

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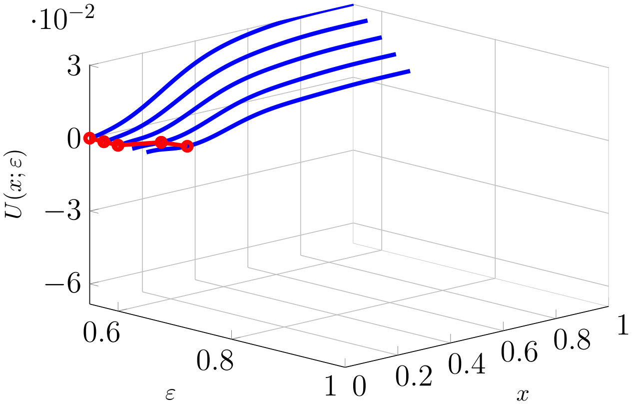

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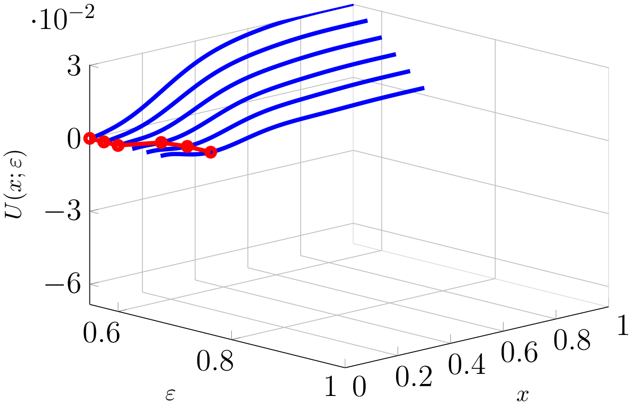

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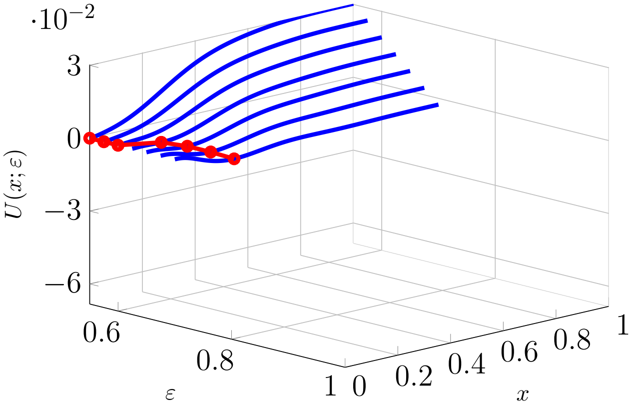

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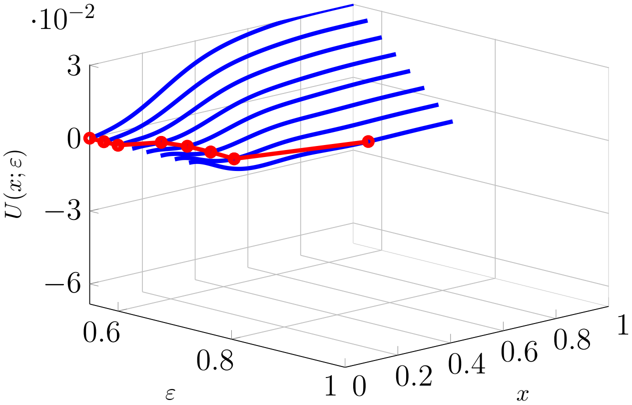

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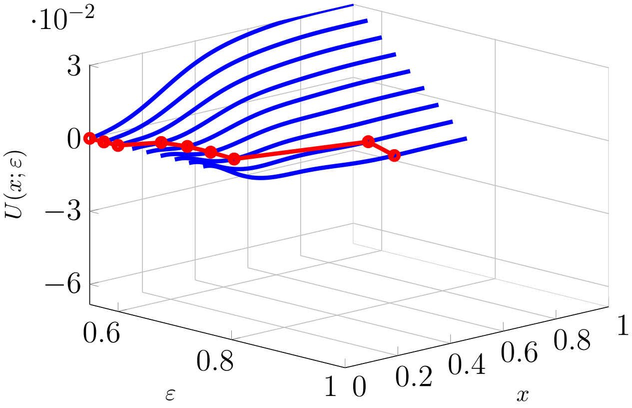

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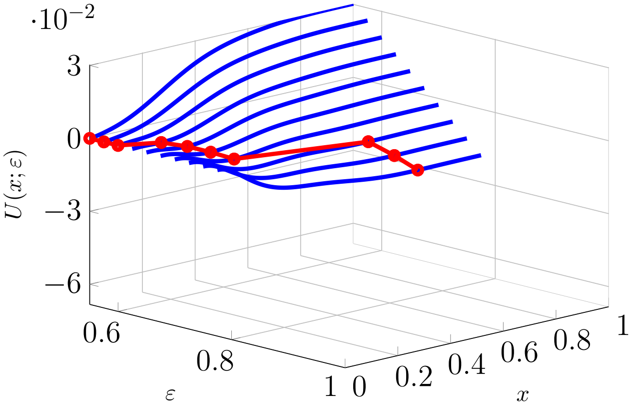

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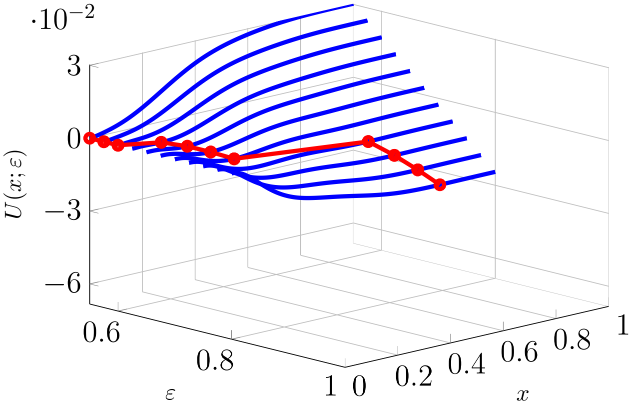

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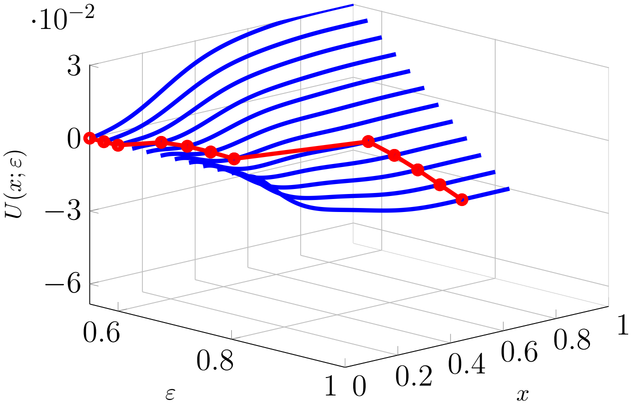

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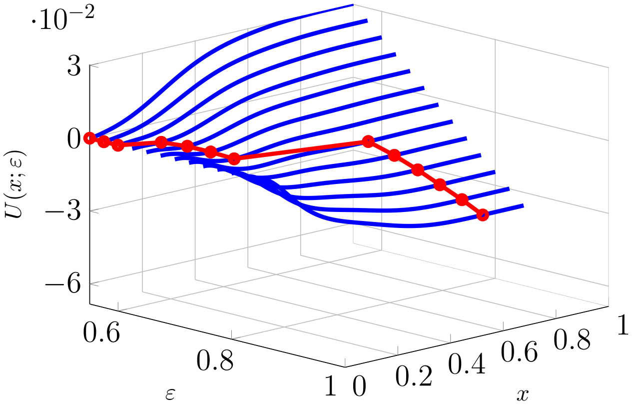

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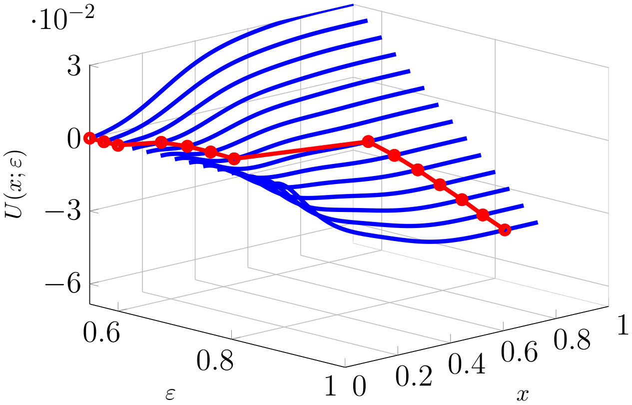

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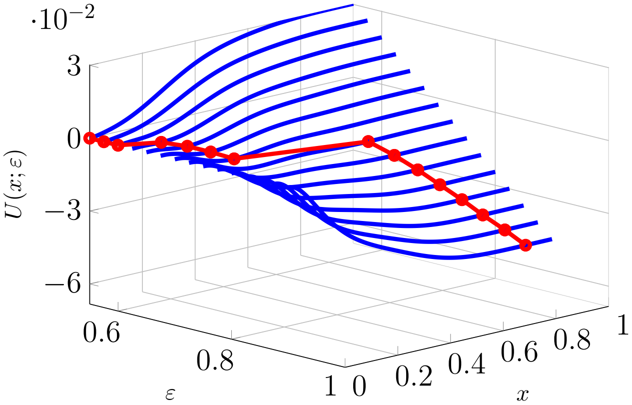

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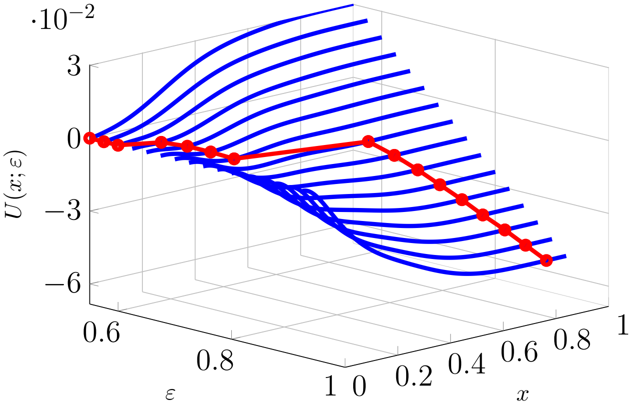

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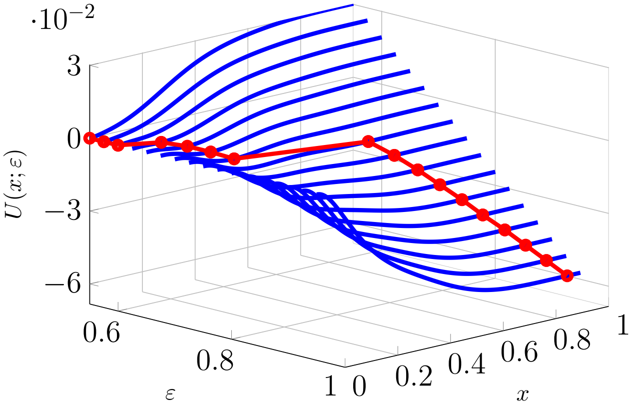

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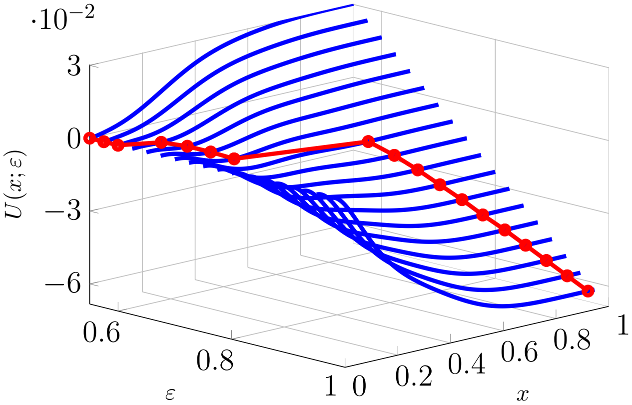

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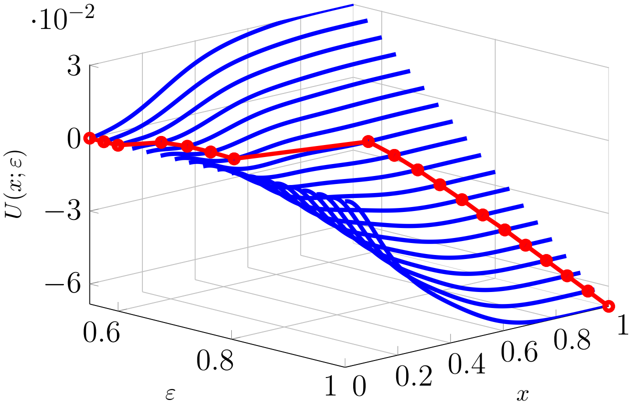

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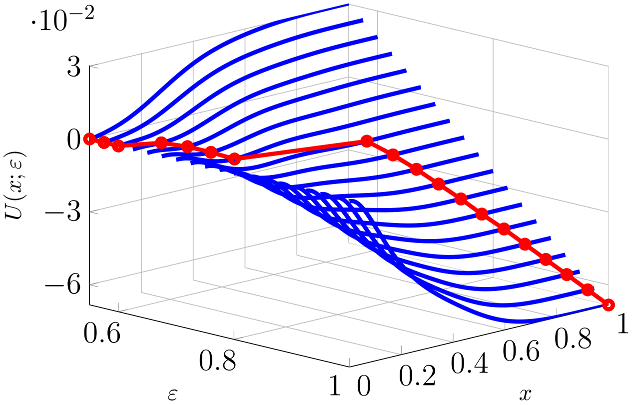

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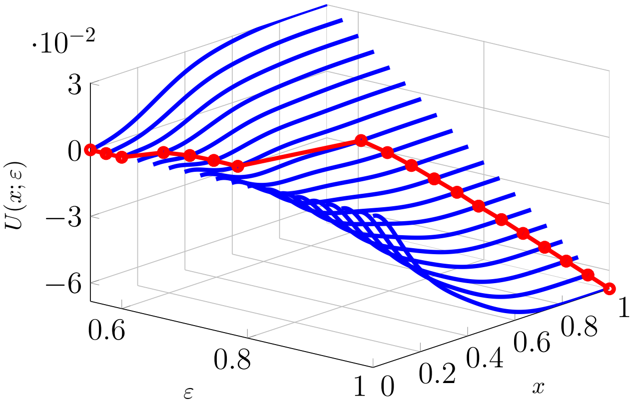

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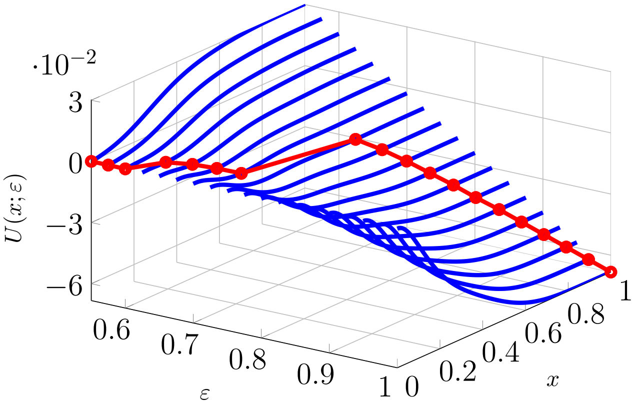

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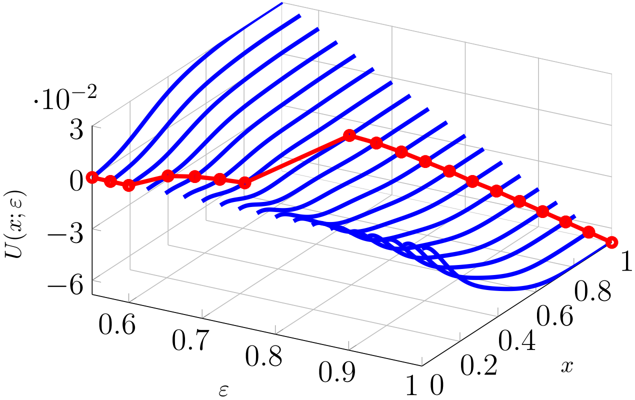

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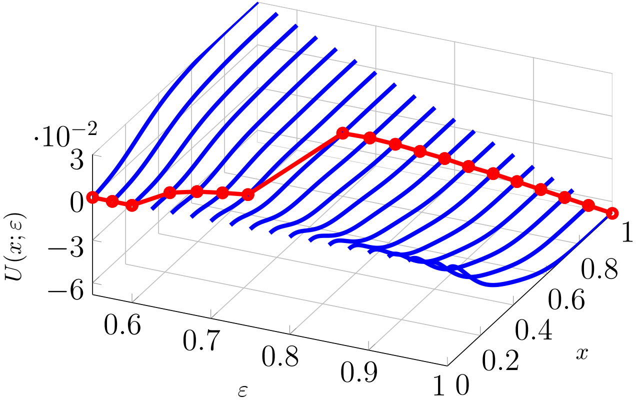

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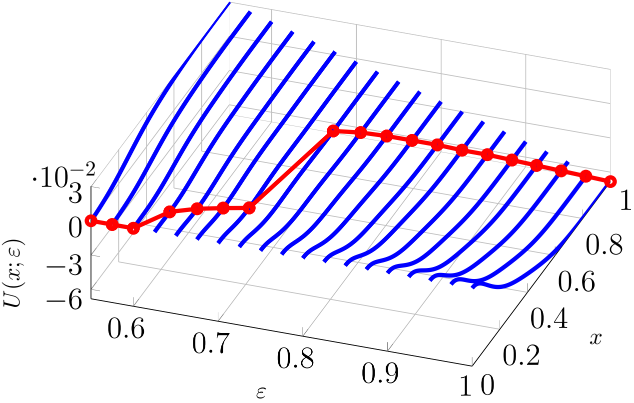

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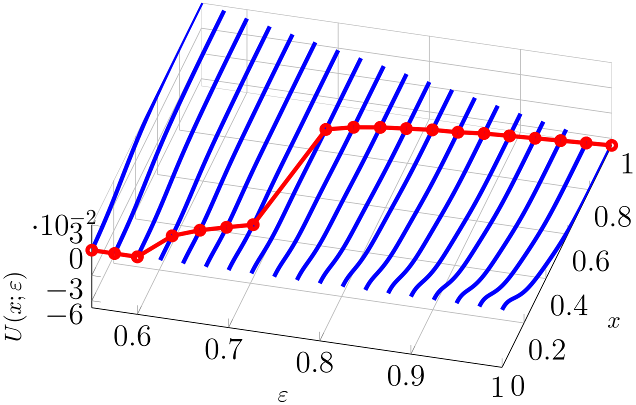

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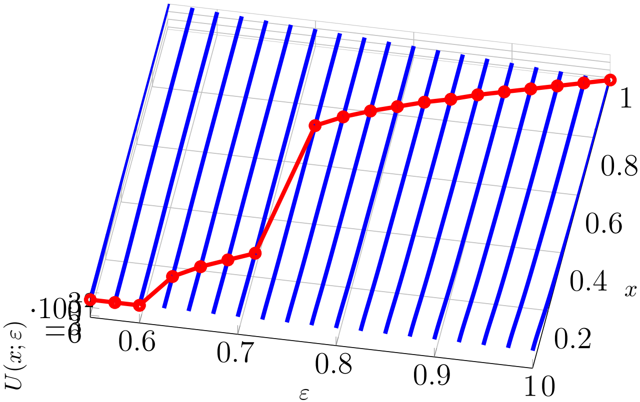

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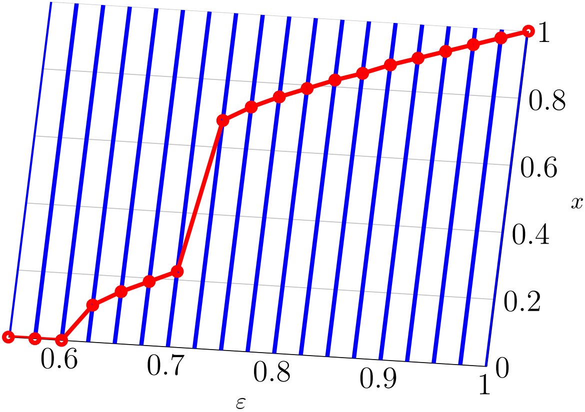
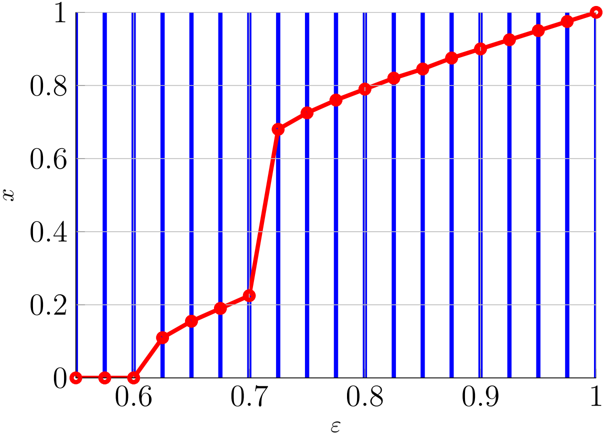
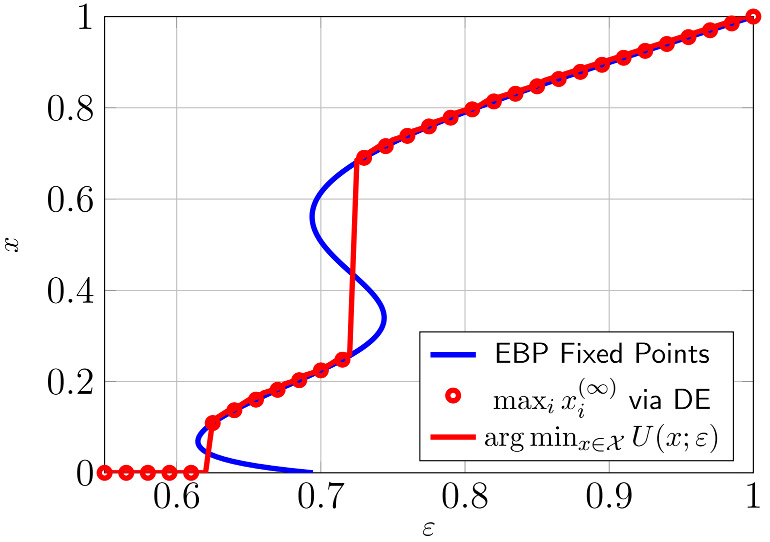


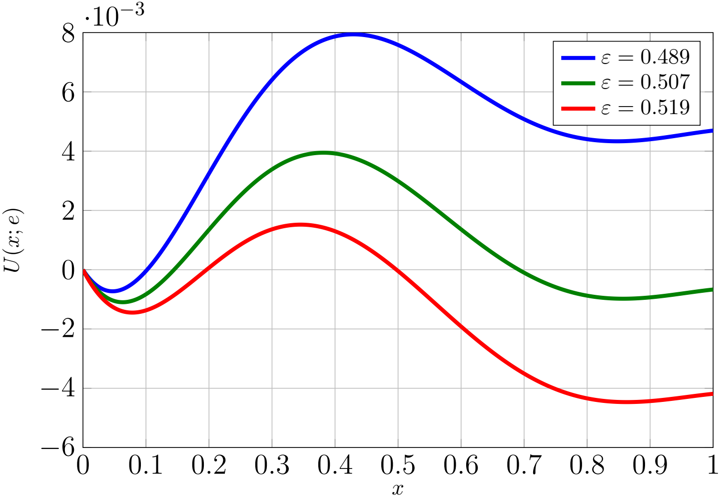
That Looks Familiar



That Looks Familiar



Example: Analysis of an LDGM Ensemble



$f(y;\varepsilon) = \lambda(y)$ $\qquad$ $\lambda(x) = x^5$

$g(x;\varepsilon) = 1 - (1-\varepsilon)\rho(1-x)$ $\qquad$ $\rho(x) = \frac{2}{45} + \frac{2}{45}x + \frac{7}{15}x^2 + \frac{4}{9}x^3$

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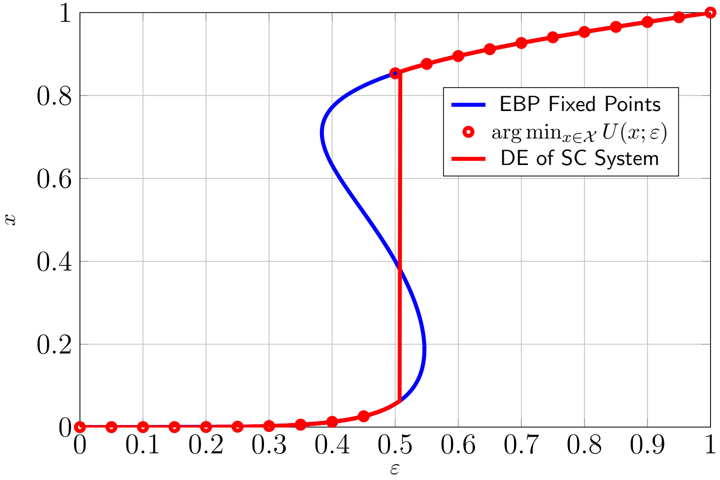

$$f(y;\varepsilon) = \lambda(y) \qquad \lambda(x) = x^5$$

$$g(x;\varepsilon) = 1-(1-\varepsilon)\rho(1-x) \qquad \rho(x) = \tfrac{2}{45} + \tfrac{2}{45}x + \tfrac{7}{15}x^2 + \tfrac{4}{9}x^3$$

Sketch of Proof for New Result

- Proof of new upper bound similar to “simple proof”
  - Only modified (i.e., one-sided) coupled recursion is changed
  - Vector values $< \overline{x}^*$ are increased to $\overline{x}^*$ after each iteration
  - Shift bound lemma refined to

$$U_{\mathrm{c}}(\boldsymbol{S}\boldsymbol{x}) - U_{\mathrm{c}}(\boldsymbol{x}) \leq U_{\mathrm{s}}(\overline{x}^*) - U_{\mathrm{s}}\left([\boldsymbol{x}]_N\right)$$

- Proof of new lower bound is based on a few observations
  - Initializing recursion to $\underline{x}^*$ lower bounds coupled fixed point
  - Iterations only decrease the potential
  - But, initial potential value implies $\max_i x_i^{(\infty)} \geq \underline{x}^* - o(1)$

Dependence on a Parameter

- Family of admissible recursions increasing in $\varepsilon \in \mathcal{E} = [0, \varepsilon_{\max}]$
  - Scalar recursion defined by $f, g : \mathcal{X} \times \mathcal{E} \to \mathcal{X}$ with

$$x^{(\ell+1)} = f\left(y^{(\ell+1)}; \varepsilon\right) \qquad y^{(\ell+1)} = g\left(x^{(\ell)}; \varepsilon\right)$$

  - Scalar potential function $U_{\mathrm{s}} : \mathcal{X} \times \mathcal{E} \to \mathbb{R}$ defined by

$$U_{\mathrm{s}}(x; \varepsilon) = x g(x; \varepsilon) - G(x; \varepsilon) - F(g(x; \varepsilon); \varepsilon)$$

  - Our new result bounds coupled fixed point as a function of ε

$$\begin{aligned} \overline{x}^*(\varepsilon) &\triangleq \max \left\{ x \in \mathcal{X} \mid U_{\mathrm{s}}(x; \varepsilon) = \Psi(\varepsilon) \right\} \\ \Psi(\varepsilon) &\triangleq \min_{x \in \mathcal{X}} U_{\mathrm{s}}(x; \varepsilon) \end{aligned}$$

The Maxwell Construction (1)

- Under mild conditions, the envelope theorem says that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Psi(\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\min_{x\in\mathcal{X}} U_{\mathrm{s}}(x;\varepsilon) \overset{a.e.}{=} U_{\mathrm{s}}^{(0,1)}\left(\overline{x}^*(\varepsilon);\varepsilon\right)$$

  - Proof sketch: derivative of minimum depends on location $\overline{x}^*(\varepsilon)$ and ε but the location term is zero due to minimum

- Computing $U_{\mathrm{s}}^{(0,1)}(x;\varepsilon) \triangleq \frac{\mathrm{d}}{\mathrm{d}\varepsilon}U_{\mathrm{s}}(x;\varepsilon)$ shows that

$$\Psi(\varepsilon) = -\int_0^{\varepsilon}\Big(G^{(0,1)}\left(\overline{x}^*(t);t\right) + F^{(0,1)}\left(g(\overline{x}^*(t);\varepsilon);t\right)\Big)\,\mathrm{d}t$$

- For LDPC codes, we get $-\frac{1}{L'(1)}$ times the MAP EXIT integral

$$\Psi(\varepsilon) = -\frac{1}{L'(1)}\int_0^{\varepsilon} L\left(1-\rho(1-\overline{x}^*(t)\right)\mathrm{d}t$$

The Maxwell Construction (2)

- The curve $\Psi(\varepsilon) = \min_{x \in \mathcal{X}} U_{\mathrm{s}}(x; \varepsilon)$ is Lipschitz continuous
- The curve $\overline{x}^*(\varepsilon)$ only jumps when the above minimum is achieved at multiple x values
- Consider two ends of a $\overline{x}^*(\varepsilon)$ jump discontinuity:
  - They must have the same value of the potential
  - If smooth fixed-point curve connects them, the integral along fixed-point curve must be zero
  - This is equivalent to the Maxwell construction

Conclusions

- We analyze coupled scalar recursions
  - Coupled fixed point given by minimizer of scalar potential
  - Extends "saturation" from threshold to Maxwell curve
  - Valuable for systems with trivial perfect decoding thresholds
    - For example, LDGM codes have $x_i^{(\infty)} \to 0$ only if $\varepsilon \to 0$
- Dependence on a parameter easily incorporated
  - Min-potential curve $\Psi(\varepsilon) = \min_{x \in \mathcal{X}} U_{\mathrm{s}}(x; \varepsilon)$ of SC system is analogous to the "negative BP conditional entropy"
  - If smooth fixed-point curve connects discontinuities, then Maxwell construction gives the $x^*(\varepsilon)$ curve, which is analogous to the Maxwell or "conjectured MAP" curve