Large Deviations for Channels with Memory

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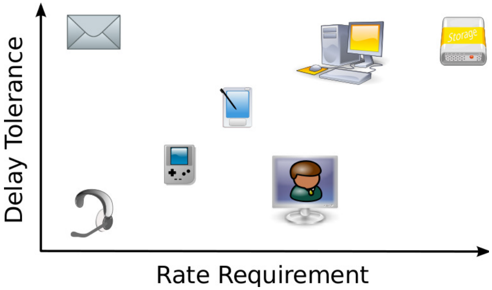
Digital Landscape

Delay Tolerance and Rate Requirement



Context

- Packetized system supporting delay-sensitive traffic
- Point-to-point communication with partial feedback

Overview

Main Objective

- Analytic framework for delay sensitive systems
- Rigorous treatment of channels with memory

Highlights

- Focus on delay sensitive systems
- Random linear codes
- Large deviations

Potential Applications

- Optimal Code Rate
- Interface Selection

Preview of Result

Main Result

- Large deviation principles on the delay and service rate for channels with memory

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Why Large deviations?

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Why Large deviations?

- Represents softer constraints on large buffers

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- Cumbersome to evaluate explicit probabilities

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- Large deviation principles on the delay and service rate for channels with memory

Why Large deviations?

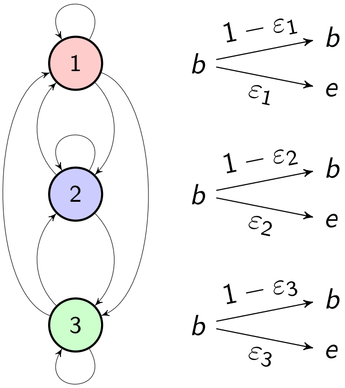
- Represents softer constraints on large buffers
- Cumbersome to evaluate explicit probabilities
- Provides foundation for other concepts like effective capacity

Channel Abstraction - Finite State Bit Erasure Channel

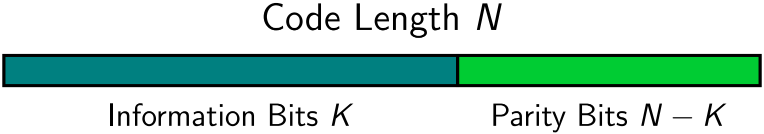
- At each instant channel can be in one of k states
- Channel transitions over time form a first-order Markov process
- In state i, transmitted bit is erased with probability ε_i
- Transition matrix

$$\begin{bmatrix} p_{11} & \cdots & p_{1k} \\ \vdots & \ddots & \vdots \\ p_{k1} & \cdots & p_{kk} \end{bmatrix}$$

Evolution at bit level



Coding Scheme



Code Generation

- Generate a binary random parity check matrix $\mathbf{H}$ of size $(N-K) \times N$
- Codebook is null space of $\mathbf{H}$

Code Properties

- Decoding failure probability (under ML) with e erasures $P_f(N-K, e) =$

$$1 - \prod_{i=0}^{e-1} \left(1 - 2^{i-(N-K)}\right)$$

Distribution of Erasures induced by Channel

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- Interested in conditional probabilities of type

$$\psi = \Pr(e \text{ Erasures}, \underbrace{C_{N+1} = j}_{\text{Channel exit state}} \mid \underbrace{C_1 = i}_{\text{Channel beginning state}})$$

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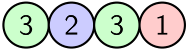

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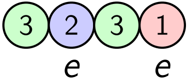

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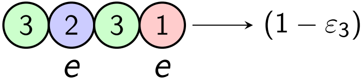

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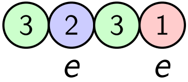

$\longrightarrow (1 - \varepsilon_3) \cdot p_{32}\varepsilon_2$

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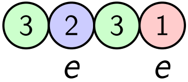

$$\longrightarrow (1-\varepsilon_3)\cdot p_{32}\varepsilon_2 \cdot p_{23}(1-\varepsilon_3)\cdot p_{31}\varepsilon_1$$

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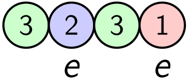
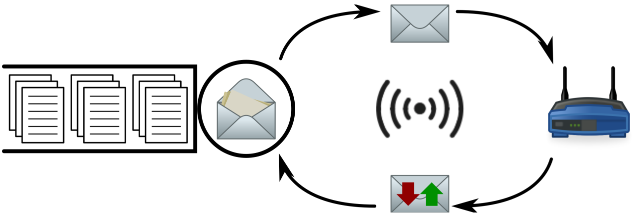


$$\longrightarrow (1-\varepsilon_3)\cdot p_{32}\varepsilon_2 \cdot p_{23}(1-\varepsilon_3)\cdot p_{31}\varepsilon_1$$

- ψ is the coefficient of x^e in $[\mathbf{P}_x^N]_{ij}$, where

$$\mathbf{P}_x = \begin{bmatrix} p_{11}(1-\varepsilon_1+\varepsilon_1 x) & \cdots & p_{1k}(1-\varepsilon_1+\varepsilon_1 x) \\ \vdots & \ddots & \vdots \\ p_{k1}(1-\varepsilon_k+\varepsilon_k x) & \cdots & p_{kk}(1-\varepsilon_k+\varepsilon_k x) \end{bmatrix}$$

Packetized Model & Acknowledgments



- Acknowledgements of successful transmissions are available
- When transmission fails, data segments are re-transmitted

Hidden Markovian System

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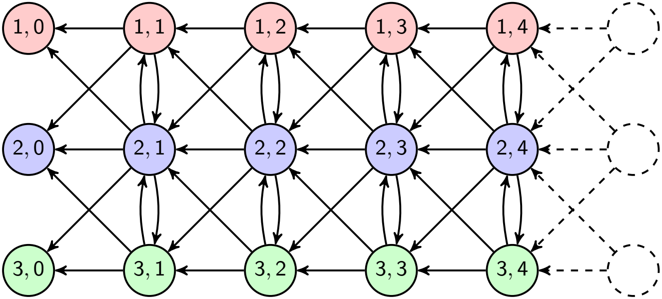
Adding Channel State to Queue Length (C_{sN+1}, q_s)



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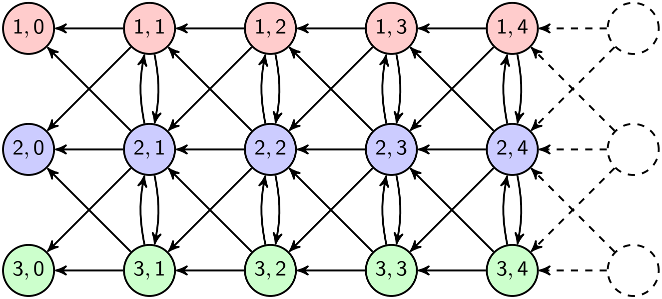
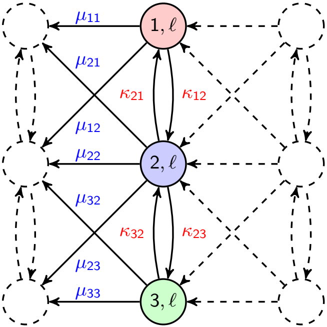


- The aggregate system is Markovian

Hidden Markovian System - System Evolution

Level Transitions



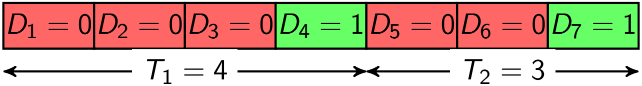
System Evolution

- Governed by Matrices: **M** (Success) and **K** (Failure)

$$\mathbf{M} = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{bmatrix}$$

Delay & Service Rate - I



Delay

- T_q is the time for delivery of data segment q
- Average Time:

$$Y_m = \frac{T_1 + \cdots + T_m}{m}$$

Service Rate

- D_s (0 or 1) indicates the success of transmission s
- Average Service Rate:

$$Z_n = \frac{D_1 + \cdots + D_n}{n}$$

Delay & Service Rate - II

Delay

- τ - delay requirement
- Minimizing $\Pr(Y_m > \tau)$ is a service guarantee on delay

Service Rate

- γ - service requirement
- Minimizing $\Pr(Z_n < \gamma)$ is a service guarantee on rate

Basis for the large deviation principles

- Look at the asymptotic decay rates (as $m, n \to \infty$) of $\Pr(Y_m > \tau)$ and $\Pr(Z_n < \gamma)$
- For large buffers large deviations characterization is appropriate

Large Deviations - A Brief Overview

- From WLLN, $\Pr\left(\frac{X_1+\cdots+X_n}{n} > \delta\right) \to 0$, when $\delta > E(X)$
- Large deviations theory characterizes the exponential decay rate with which $\Pr \to 0$

$$\underbrace{I(\delta)}_{\text{Rate function}} = \lim_{n\to\infty} -\frac{1}{n}\log\Pr\left(\frac{X_1+\cdots+X_n}{n} > \delta\right)$$

- When $\{X_i\}$ are iid, Cramér's Theorem characterizes the large deviations principle (LDP)
- When $\{X_i\}$ are weakly dependent, invoke Gärtner-Ellis Theorem

Two Deviations

Theorem

Y_m satisfies a large deviations principle with rate function $I_{del}(\cdot)$ where

$$I_{del}(x) = \sup_{\lambda \in \mathbb{R}} \{\lambda x - \log \rho((\mathbf{I} - \mathbf{K}e^{\lambda})^{-1} \mathbf{M} e^{\lambda})\}$$

Theorem

Z_n satisfies a large deviations principle with rate function $I_{ser}(\cdot)$ where

$$I_{ser}(x) = \sup_{\lambda \in \mathbb{R}} \{\lambda x - \log \rho(\mathbf{K} + \mathbf{M} e^{\lambda})\}$$

- $\rho(\cdot)$ is the spectral radius of a Matrix

Relation between the deviations

- Set equivalence

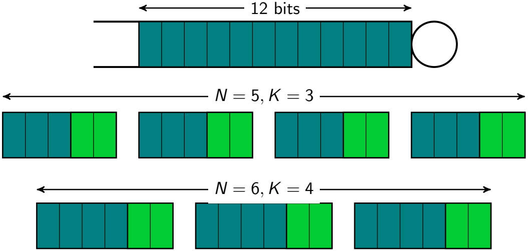
$$\{T_1 + \cdots + T_m > n\} = \{D_1 + \cdots + D_n < m\}$$

- Relation

$$xI_{del}(1/x) = I_{ser}(x)$$

This implies the means (in the asymptotic sense) of $\{D_s\}$ and $\{T_q\}$ have an inverse relation

Scaling for Commonality



Normalized Delay

- $\tilde{Y}_m = \frac{NT_1 + \cdots + NT_{m/K}}{m}$
- $\frac{1}{K} I_{del}(\frac{K}{N}\tau)$

Normalized Service Rate

- $\tilde{Z}_n = \frac{KD_1 + \cdots + KD_{n/N}}{n}$
- $\frac{1}{N} I_{ser}(\frac{N}{K}\gamma)$

GSM Example

System Parameters

Speech frame length	228 bits
Physical layer	1 bit in $40\mu s$
Delay Tolerance	$40ms$

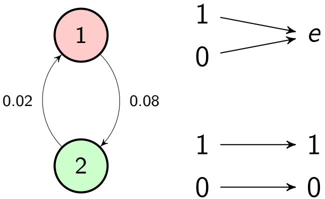
- Requires 228 bits in 1000 channel uses
- Choose $\tau = 1000/228$ for the example

Numerical Analysis

System Parameters

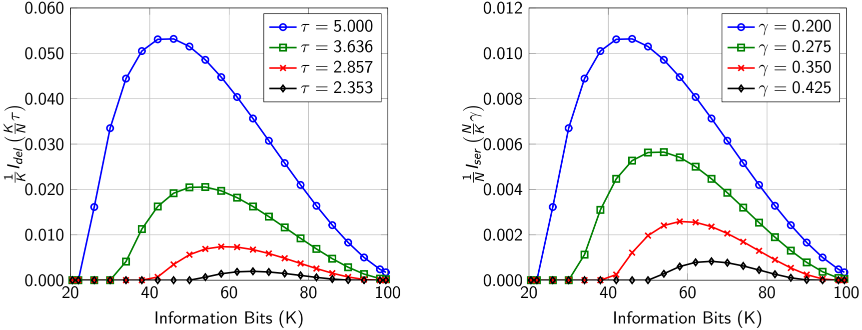
Channel	Gilbert-Elliott Channel
Erasure Probabilities	$\varepsilon_1 = 1, \varepsilon_2 = 0$
Channel Memory	0.9
Code length	N=114

Channel Diagram



- Average erasure rate is 20%
- K is the parameter to be optimized

Numerical Analysis



Optimal Code Rate

- Depends heavily on the underlying traffic
- Very conservative as the service requirements are lowered
- The perspective of delay and service rate are related

Concluding Remarks

Main Contributions

- Provided a methodology for designing delay sensitive systems
- Rigorously modeled channels with memory

Possible Extensions

- Channel Estimation
- Analyze variable block lengths