A Rate-Distortion Perspective on Multiple Decoding of Reed-Solomon Codes

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Brief background on Reed-Solomon (RS) codes

- RS codes are one of the most widely used error-correcting codes in digital communication and data storage.

- \((n, k)\) RS codes have rate \(\frac{k}{n}\) and minimum distance \(d_{\text{min}} = n - k + 1\) which is the best possible (MDS).

- Can correct both random and burst errors.

- Efficient conventional hard-decision (HD) decoding algorithms such as Berlekamp-Massey (BM) can correct up to half \(d_{\text{min}}\).
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  - Guruswami-Sudan (GS) - can correct errors beyond half $d_{\text{min}}$.
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Motivation

- Multiple runs of some low complexity algorithm.

- For example: multiple runs of error-and-erasure decoding (BM algorithm), each time with a different set of erasure patterns.

Definition

- \( \hat{x}^n \in \mathbb{Z}_2^n \triangleq \{0,1\}^n \) as an erasure pattern: at index \( i \),
  \[
  \hat{x}_i = \begin{cases} 
  0, & \text{if symbol is erased} \\
  1, & \text{if symbol is not erased} 
  \end{cases}
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- \( x^n \in \mathbb{Z}_2^n \) as an error pattern: at index \( i \),
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- Sort the codeword positions in increasing reliability order.

- Generalized Minimum Distance (GMD) decoding [Forney]
  Repeat error-and-erasure decoding while successively erasing an even number of the least reliable positions (LRPs).

- Successive Error-and-erasure Decoding (SED) [Lee-Kumar]
  SED($l, f$) repeats error-and-erasure decoding with every combination of an even number $\leq f$ of erasures within the $l$ LRPss.

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Error pattern
001011…

Erasure patterns of GMD
111111…
001111…
000011…

Erasure patterns of SED(3, 2)
111111…
001111…
010111…
100111…
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Main results

- Design a R-D framework to analyze the asymptotic trade-off between performance vs complexity of multiple error-and-erasure decoding.
  - The framework is extended to analyze multiple ASD decoding.
  - Propose a family of multiple-decoding algorithms are that achieve better performance-vs-complexity trade-off than other algorithms.
  - The algorithm that achieves the best trade-off uses the set of patterns generated by random codes combined with covering codes.
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Multiple error-and-erasure decoding

Single error-and-erasure decoding threshold

Consider an \((n, k)\) RS code. If \(e\) symbols are erased, the BM algorithm can correct \(v\) errors in unerased positions if \(2v + e < n - k + 1\).

- A multiple error-and-erasure decoding is considered to succeed if the decoding threshold is satisfied for at least one round of decoding.

- **Our idea**: connect to a R-D (covering) problem where the multiple-decoding succeeds if the error pattern \(\blacksquare\) is covered by at least one ball centered at an erasure pattern \(\bullet\).
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### Distortion measure

**Definition**

Given a *letter-by-letter* distortion measure $\delta$, the distortion between and error pattern $x^n$ and an erasure pattern $\hat{x}^n$ is $d(x^n, \hat{x}^n) = \sum_{i=1}^{n} \delta(x_i, \hat{x}_i)$.

**Proposition**

If we choose $\delta : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \rightarrow \mathbb{R}_{\geq 0}$ as follows

<table>
<thead>
<tr>
<th>Erasure</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes (0)</td>
<td>No (1)</td>
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then $2v + e < n - k + 1$ reduces to the form $d(x^n, \hat{x}^n) < n - k + 1$.

**Question:** How many decoding attempts are needed to achieve a fixed distortion threshold?
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Our approach

- **Problem statement**: Build a set $\mathcal{B}$ of no more than $2^R$ erasure patterns $\hat{x}^n$ in order to

$\max_{\mathcal{B}:|\mathcal{B}| \leq 2^R} \Pr\{ \min_{\hat{x}^n \in \mathcal{B}} d(x^n, \hat{x}^n) < n - k + 1 \}.$

- **Asymptotic solution 1**:  
  - View the error pattern $x^n$ as a source sequence and the erasure pattern $\hat{x}^n$ as a reproduction sequence $\rightarrow$ a source coding problem.
  - R-D theory: the set $\mathcal{B}$ of $2^R$ reproduction sequences (erasure patterns) can be generated randomly according to $q$ so that

  $\lim_{n \to \infty} E \left[ \min_{\hat{x}^n \in \mathcal{B}} d(x^n, \hat{x}^n) \right] \leq D$

  - Thus, for large enough $n$, with high probability we have

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Proposed general multiple-decoding algorithm

- **Phase I**: Compute rate-distortion function (run one time)
  - **Step 1**: Empirically compute the reliability matrix $[P_{1}^{(t)}]_{j,i}$ during time $t$ for $t = 1, \ldots, \tau$.
  - **Step 2**: Sort the probabilities in increasing order reliability order of codeword positions and get an average matrix $\bar{P}$ over all time $t$.
  - **Step 3**: Compute the R-D function using probability matrix $\bar{P}$. Determine the point on the R-D curve that corresponds to a designated rate $R$ along with the test-channel input-distribution $q$ that achieves that point.
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Proposed general multiple-decoding algorithm (cont.)

- **Phase II**: Run actual decoder.
  - **Step 4**: Based on the actual received signal sequence, determine the permutation $\sigma$ that gives the reliability order of codeword positions.
  - **Step 5**: Randomly generate a set of $2^R$ erasure patterns using the test-channel input-probability distribution vector $q$ and permute the indices of each erasure pattern by the permutation $\sigma^{-1}$.
  - **Step 6**: Run multiple attempts of the corresponding decoding scheme (e.g. error-and-erasure decoding) using the set of erasure patterns in Step 5 to produce a list of candidate codewords.
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  - **Step 5**: Randomly generate a set of $2^R$ erasure patterns using the test-channel input-probability distribution vector $q$ and permute the indices of each erasure pattern by the permutation $\sigma^{-1}$.
  - **Step 6**: Run multiple attempts of the corresponding decoding scheme (e.g. error-and-erasure decoding) using the set of erasure patterns in Step 5 to produce a list of candidate codewords.
  - **Step 7**: Use Maximum-Likelihood (ML) decoding to pick the best codeword on the list.
Generalizations and extensions

- This general R-D framework can be extended in two ways:
  - Generalize error and erasure patterns in multiple error-and-erasure decoding to make better use of the soft information.
  - Analyze multiple ASD decoding.
- The trick is to find an appropriate distortion measure $\delta$ to convert the decoding threshold to the form $d(x^n, \hat{x}^n) < D$.
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Generalized multiple error-and-erasure decoding

Definition
Consider a positive integer $l$.

- $x^n \in \mathbb{Z}_{l+1}^n$ as a generalized error pattern: at index $i$,
  $$x_i = \begin{cases} 
  0, & \text{none of the first } l \text{ most likely symbols is correct} \\
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- $\hat{x}^n \in \mathbb{Z}_{l+1}^n$ as a generalized erasure pattern: at index $i$,
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Theorem
We choose $\delta : \mathbb{Z}_{l+1} \times \mathbb{Z}_{l+1} \rightarrow \mathbb{R}_{\geq 0}$ defined by $\delta(x, \hat{x}) = [\Delta]_{x, \hat{x}}$ in terms of

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\vdots & \vdots & \ddots & \vdots & \vdots \\
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\end{pmatrix} \quad \Rightarrow \quad \text{decoding threshold becomes} \quad d(x^n, \hat{x}^n) < n - k + 1.$$


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**Split-covering approach**

- For a finite $n$, the random coding approach may have problems with only a few LRPss.
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  Sort the RS c.w. positions in increasing reliability order.

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Simulation results

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