On the Queueing Behavior of Random Codes over a Gilbert-Elliott Erasure Channel

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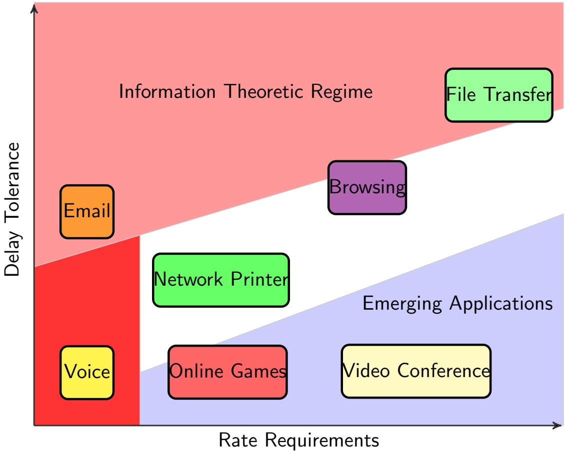
Digital Landscape

Delay Rate and Rate Requirements

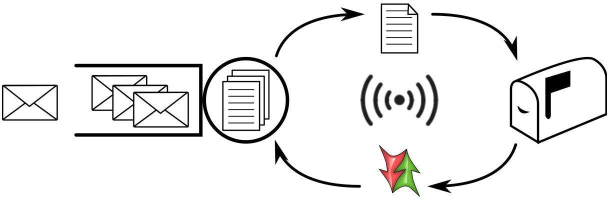


Potential Scenarios

- Ad-hoc/sensor networks
- Mesh networks
- Cellular networks

Conceptual Model

Three-Component System

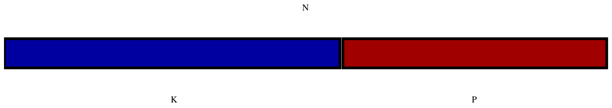


Context

- Point-to-point communication with limited feedback
- Reliability achieved through error-control coding
- Packetized system supporting delay-sensitive traffic
- Finite block-length codes

Coding Scheme

Random Binary Code



Random Coding

- Generate random parity-check matrix $H \in \{0,1\}^{N-K \times N}$
- Maximum likelihood decoding

Coding versus Queueing

- Fundamental block-length N chosen suitably for applications
- High code-rate $\Rightarrow$ frequent retransmission
- Low code-rate $\Rightarrow$ low throughput

Problem Statement

Question

Find optimal code-rate K/N for best queueing performance, given a finite block-length code employing random binary coding over a Gilbert-Elliott erasure channel

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Preview of Results

Optimal code-rate for a certain delay-performance

- Depends heavily on the channel memory

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Preview of Results

Optimal code-rate for a certain delay-performance

- Depends heavily on the channel memory
- Can be higher than the channel capacity
- Reaches unity for highly correlated channel

Physical Layer

Correlated Erasure Channel

Channel can be in two states, under which information is lost

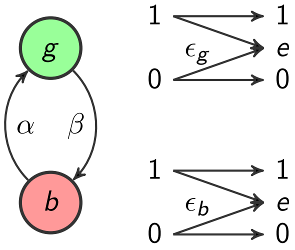
- Good state g with probability ϵ_g
- Bad state b with probability ϵ_b

Discrete-time Markov Model

- Channel state sequence $\{C_n\}$ is discrete-time Markov process
- Stationary probability transition matrix

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

Channel Diagram



Code Performance

Probability of Decoding Failure

For a random binary codeword of length N with $N - K$ parity bits and E erasures

$$P_{\mathrm{f}}(N-K, E) = 1 - \prod_{i=0}^{E-1} \left(1 - 2^{i-(N-K)}\right)$$

Distribution of Erasures

- Decoding depends on number of erasures E within codeword
- E depends on number of visit to either channel-states
- Interested in conditional probabilities of type

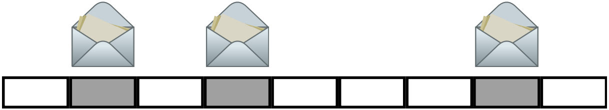
$$\Pr(E = e, C_{N+1} | C_1)$$

where C_n is the channel-state for bit n

Packet Arrivals

Bernoulli arrival with random length

Arrival Process



- Sampling period N, at onset of every codeword transmission
- IID Bernoulli arrival of a packet of random IID length

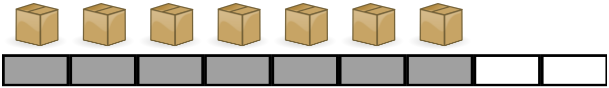
Packet Length Distribution

- IID geometrically distributed L number of bits in each packet
- Each packet fragmented into M segment of K bits

Packet Departures

Departures depend on Gilbert-Elliott channel and code-rate K/N

Data Fragments

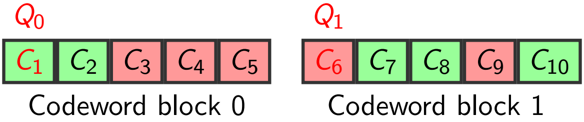


- Number of codewords per packet is $M = \lceil L/K \rceil$
- M is also geometric,and hence memoryless

Departure of Head-Packet from Queue

- Successful decoding of most recent codeword
- Most recent codeword carries the final segment of information
- Instantaneous feedback of codeword reception

Aggregate Model and Transition Probabilities



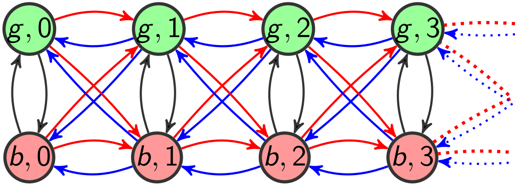
Queue-length measure in terms of number of packets

- Queue length at onset of block s is Q_s
- Channel state at same instant C_{sN+1}
- State of augmented Markov state is

$$U_s = (C_{sN+1}, Q_s)$$

- Queue-length hidden Markov, but aggregate system Markov

State Diagram of Aggregate System



- Up-transitions λ, self-transitions κ, and down-transitions μ

$$\mathbf{A}_0 = \begin{bmatrix} \lambda_{bb} & \lambda_{bg} \\ \lambda_{gb} & \lambda_{gg} \end{bmatrix} \quad \mathbf{A}_1 = \begin{bmatrix} \kappa_{bb} & \kappa_{bg} \\ \kappa_{gb} & \kappa_{gg} \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} \mu_{bb} & \mu_{bg} \\ \mu_{gb} & \mu_{gg} \end{bmatrix}$$

- Transition probabilities λ, κ and μ depend on
  - Probability of decoding failure
  - Bernoulli parameter for arrival
  - Distribution of erasures via Gilbert-Elliott channel

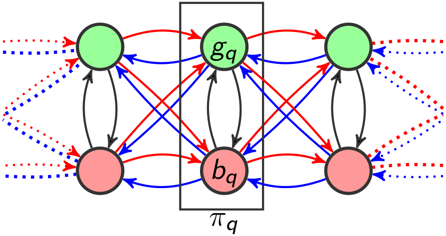
Chapman-Kolmogorov Equations

For stable system, equilibrium distribution is $\pi = (\pi_0, \pi_1, \pi_2, \dots,)$ where

$$\pi_q \triangleq \lim_{s\to\infty} [\Pr(U_s = (b, q)) \quad \Pr(U_s = (g, q))]$$

is unique solution to balance equations (plus boundary conditions)

$$\pi_q = \pi_{q-1}\mathbf{A}_0 + \pi_q\mathbf{A}_1 + \pi_{q+1}\mathbf{A}_2, \quad q \geq 2$$



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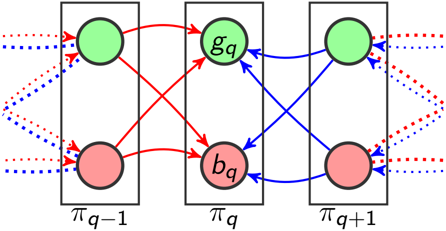

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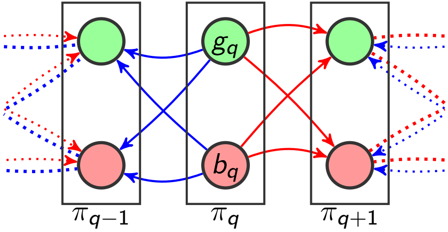


Matrix Geometric Method

Theorem

Let the positive matrix $\mathbf{R}$ *be defined as the limit, starting from* $\mathbf{R}_0 = \mathbf{0}$, *of the matrix recursion*

$$\mathbf{R}_{j+1} = (\mathbf{A}_0 + \mathbf{R}_j^2 \mathbf{A}_2)(\mathbf{I} - \mathbf{A}_1)^{-1}.$$

Then, the qth-level stationary distribution π_q *satisfies* $\pi_{q+1} = \pi_1 \mathbf{R}^q$ *for* $q \geq 1$ *with* $\pi_1 = \pi_0 \mathbf{Z}$ *from boundary conditions.*

This provides us an efficient way of computing equilibirum queue-distribution in terms of code-rate, arrival and channel parameters.

System Parameters

Envisioned Scenario

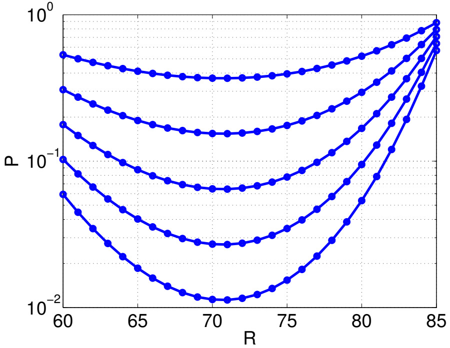
Parameters selected to mimic operation of pico-cell aggregator for sensor network with GSM link

$N = 114$	Code length
4.615 ms	Duration of transmission cycle
$\gamma = 0.25$	Probability of new packet arrival
$\rho = 1/175$	Mean packet size is 175 bits
9.48 Kbps	Average arrival rate

Gilbert-Elliott Channel

- Parameters are $\alpha = 0.08$ and $\beta = 0.02$
- Probability of erasure in good and bad states $\epsilon_g = 0, \epsilon_b = 1$
- Expected bit-erasure probability is 0.2, capacity is 0.8
- Channel memory decays at rate $(1 - \alpha - \beta) = 0.9$

Optimal Code Rate Selection

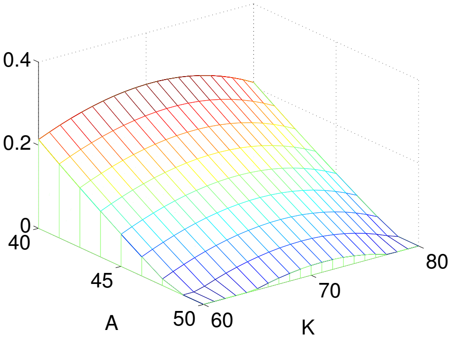


- $\Pr(Q > \tau)$ for thresholds $\tau \in \{5, 10, 15, 20, 25\}$
- Minimum represents optimal operating point
- Uniformly achieved at $K = 71$
- Best code rate is 0.62, capacity is 0.8

Property I

Optimal code rate appears robust to threshold value

Tail Decay Rate and Arrival Process



- Tail decay rate

$$-\lim_{\tau \to \infty} \frac{1}{\tau} \log \Pr(Q \geq \tau)$$

- Average arrival rate γ/ρ_r per cycle
- Maximum denotes optimal operating point

Property II

Optimal code rate seems robust to parameters of arrival process

Channel Parameters

$1-\alpha-\beta$	K^*	$\min \Pr(Q > 5)$
0	80	0.049
0.5	76	0.085
0.9	71	0.368
0.98	87	0.598
0.99	114	0.246

- Channel memory affects optimal code rate
- It impacts queue distribution
- Limiting regime becomes packet erasure channel with memory

Property III

Optimal code rate is very sensitive to channel memory

Discussion and Concluding Remarks

Main Contributions

- Integration of coding and queueing techniques to study communication for delay-sensitive traffic
- Correlated erasure channel provides insight into optimal code rate selection for short block lengths
- Optimal code rate appears linked to ratio between codeword time and coherence time

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Avenues of Future Research

- Extend results to finite-state erasure channels
- Use framework to validate/invalidate common assumptions (fluid models, instantaneous capacity, etc.)
- Binary symmetric channels with memory
- Varibale block lengths

Thank You

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